

# The algebra of the parallel endomorphisms of a germ of pseudo-Riemannian metric

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**Abstract.** On a (pseudo-)Riemannian manifold  $(\mathcal{M}, g)$ , some fields of endomorphisms *i.e.* sections of  $\text{End}(T\mathcal{M})$  may be parallel for  $g$ . They form an associative algebra  $\mathfrak{e}$ , which is also the commutant of the holonomy group of  $g$ . We study it. As any associative algebra,  $\mathfrak{e}$  is the sum of its radical and of a semi-simple algebra  $\mathfrak{s}$ . We show the following:  $\mathfrak{s}$  may be of eight different types, including the generic type  $\mathfrak{s} = \mathbb{R}\text{Id}$ , and the Kähler and hyperkähler types  $\mathfrak{s} \simeq \mathbb{C}$  and  $\mathfrak{s} \simeq \mathbb{H}$ . For  $N$  any self adjoint nilpotent element of the commutant of  $\mathfrak{s}$  in  $\text{End}(T\mathcal{M})$ , the algebra  $\mathfrak{s} \times \langle N \rangle$  is the algebra of the parallel endomorphisms of a non empty set of germs of metrics, that we parametrise. Generically, the holonomy group of these metrics is the commutant of  $N$  in the commutant  $H_{\mathfrak{s}}$  of  $\mathfrak{s}$  in  $\text{SO}^0(g)$ . To prove it, we introduce an analogy with complex Differential Calculus, the ring  $\mathbb{R}[X]/(X^n)$  replacing the field  $\mathbb{C}$ . This describes totally the local situation when the radical of  $\mathfrak{e}$  is principal and consists of self adjoint elements. We add a glimpse on the case where this radical is not principal, and give the constraints imposed to the Ricci curvature when  $\mathfrak{e}$  is non trivial.

**Keywords:** Pseudo-Riemannian, Kähler, hyperkähler, parakähler metrics, holonomy group, parallel endomorphism, nilpotent endomorphism, commutant, Ricci curvature.

**M.S.C. 2010:** 53B30, 53C29, secondary 53B35, 53C10, 53C12.

We investigate here what are the possible algebras of parallel endomorphism fields, for a germ of (pseudo-)Riemannian metric. Our motivation is the following.

**Motivation.** A Kähler metric  $g$  on some manifold  $\mathcal{M}$  may be defined as a Riemannian metric admitting an almost complex structure  $J$  which is parallel:  $DJ = 0$  with  $D$  the Levi-Civita connection of  $g$ . A natural question is to ask whether other fields of endomorphisms, *i.e.* sections of  $\text{End}(T\mathcal{M})$ , than such a  $J$ , may be parallel for a Riemannian metric. The answer is nearly immediate. First, one restricts the study to metrics that do not split into a non trivial Riemannian product, called here “indecomposable”. Indeed else, any parallel endomorphism field is the direct sum of parallel such fields on each factor (considering as a unique factor the possible flat factor). Then a brief reasoning ensures that only three cases occur:  $g$  may be generic *i.e.* admit only the homotheties as parallel endomorphisms, be Kähler, or be hyperkähler *i.e.* admit two (hence three) anticommuting parallel complex structures. The brevity of this list is due to a simple fact: the action of the holonomy group  $H$  of an indecomposable Riemannian metric is irreducible *i.e.* does not stabilise any proper subspace. In particular, this compels any parallel endomorphism field to be of the form  $\lambda\text{Id} + \mu J$  with  $J$  some parallel, skew adjoint almost complex structure. Now, such irreducibility fails in general for an indecomposable pseudo-Riemannian metric, so that a miscellany of other parallel endomorphism fields may appear. This gives rise to the question handled here:

Which (algebra of) parallel endomorphism fields may a pseudo-Riemannian metric admit ?

Its first natural step, treated here, is local *i.e.* concerns germs of metrics.

The interest of this question is also the following. When studying the holonomy of indecomposable pseudo-Riemannian metrics, the irreducible case may be exhaustively treated:

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the full list of possible groups, together with the corresponding spaces of germs of metrics (and possibly compact examples) may be provided. After a long story that we do not recall here, this has been done, even in the non metric cases, see *e.g.* the surveys [11, 21]. Yet, in the general case, the representation of  $H$  may be not semi-simple and such an exhaustive answer is out of reach, except perhaps in very low dimension, see *e.g.* the already long list of possible groups in dimension four in [2, 13]. Thus, intermediate questions are needed: not aiming at the full classification, but still significant. Investigating the commutant  $\text{End}(T_m\mathcal{M})^H$  of  $H$  at some point  $m$  of  $\mathcal{M}$ , instead of  $H$  itself — that is to say studying the algebra of parallel endomorphisms — is such a question. It has been partially treated, namely for an individual self adjoint endomorphism with a minimal polynomial of degree 2, by G. Kručković and A. Solodovnikov [16]. I thank V. S. Matveev for this reference. One may also notice that determining all the parallel tensors, not only the endomorphisms, would mean determining the algebraic closure of the holonomy group  $H$ . In that sense, studying the parallel endomorphisms is a step in a possible more general way to understand  $H$ .

Finally, looking at the invertible self adjoint elements of  $\text{End}(T_m\mathcal{M})^H$  is exactly determining all the pseudo-Riemannian metrics sharing the same Levi-Civita connection, which is also a useful question. The skew adjoint ones are linked with the parallel symplectic forms.

So we investigate here the algebra of the title, the interest of the work being that:

We deal with indecomposable metrics the holonomy of which is never supposed to be irreducible or totally reducible.

As it is classical in holonomy problems, the present question is twofold: **(i)** which algebras are possible ? **(ii)** By which sets of germs of metrics are they produced ? We will handle both, we say just below to what extent.

A last motivation, linked with **(ii)**, is the following. A metric makes some endomorphism field  $U$  parallel if and only if  $U \in \text{End}(T_m\mathcal{M})^H$ . So studying such metrics may lead to produce the commutant of  $U$  in some classical holonomy group, as a holonomy group itself. This would be new in general for  $U$  nilpotent. An example of metric having such a commutant, namely in  $H_0 = \text{SO}^0(p, q)$ , as its holonomy group has been recently built by A. Bolsinov and D. Tsonev [6]. We parametrise sets of such metrics, see just below how.

**Contents and structure of the article.** As any associative algebra,  $\text{End}(T_m\mathcal{M})^H$  classically splits into a semi-direct product  $\mathfrak{s} \ltimes \text{Rad}(\text{End}(T_m\mathcal{M})^H)$  with  $\mathfrak{s}$  semi-simple and  $\mathfrak{n} := \text{Rad}(\text{End}(T_m\mathcal{M})^H)$  a nilpotent ideal, its radical. Once again, only the semi-simple part  $\mathfrak{s}$  allows an exhaustive treatment, provided here, whereas no list of possible forms for  $\mathfrak{n}$  may be given. Recall that, purely algebraically, the classification of nilpotent associative algebras is presently out of reach. Even the case of *pairs* of *commuting* nilpotent matrices is an active subject; we did not find any explicit review of it, but *e.g.* [1] and its bibliography may be consulted. Besides, our geometric context does not seem to simplify significantly the algebraic nature of  $\mathfrak{n}$ . So we treat here a natural first step: the case where  $\mathfrak{n}$  is principal *i.e.* generated by a unique element. Our main statement may be summed up as follows.

**Theorem** *The semi-simple part  $\mathfrak{s}$  of the algebra  $\text{End}(T_m\mathcal{M})^H$  may be of eight different types, including the generic type  $\mathfrak{s} = \mathbb{R} \text{Id}$ . For  $N$  any self adjoint nilpotent — possibly null — element of the commutant of  $\mathfrak{s}$  in  $\text{End}(T_m\mathcal{M})$ , the algebra  $\mathfrak{s} \ltimes \langle N \rangle$  is the algebra of parallel endomorphisms of a non empty set of germs of metrics (that we parametrise). Generically, the holonomy group of these metrics is the commutant of  $N$  in the commutant  $H_{\mathfrak{s}}$  of  $\mathfrak{s}$  in  $\text{SO}^0(g)$ .*

This result allows a glimpse on the general case for  $\mathfrak{n}$ . Finally we describe the consequences of the existence of parallel endomorphisms on the Ricci curvature.

The article is divided into five parts.

- Part 1 is essentially devoted to  $\mathfrak{s}$ . We introduce the decomposition  $\text{End}(T_m\mathcal{M})^H = \mathfrak{s} \ltimes \text{Rad}(\text{End}(T_m\mathcal{M})^H)$  in §1.1 and some natural objects associated with a *reducible* holonomy representation in §1.2; we give in Proposition 1.8 a simple but very noticeable commutation property in  $\text{End}(T_m\mathcal{M})^H$ . We list in §1.3, Theorem 1.11 and Tables 1 and 2, the eight possible forms of  $\mathfrak{s}$ , with two corollaries. We give the corresponding spaces of germs of metrics in §1.4.

- Part 2 is devoted to  $\mathfrak{n} := \text{Rad}(\text{End}(T_m\mathcal{M})^H)$ , in the sense that it parametrises the set  $\mathcal{G}$  of germs of metrics  $g$  admitting a parallel nilpotent endomorphism field  $N$ . At least, it deals with the case where  $N$  is  $g$ -self adjoint. Indeed if some  $N \in \mathfrak{n}$  is parallel, so are its self- and skew-adjoint parts  $\frac{1}{2}(N \pm N^*)$ , so it is natural to study first the cases  $N^* = \pm N$ . It turns out that a natural local description of  $\mathcal{G}$  follows from an analogy with complex differential calculus,  $N$  replacing  $J$  and  $\mathbb{R}[X]/(X^n)$  replacing  $\mathbb{C} = \mathbb{R}[X]/(X^2 + 1)$ . Counterparts of holomorphic functions, of their power series expansion, appear. So we introduce this analogy in §2.1 and §2.2, notably with Definition 2.8 and Proposition 2.16. Then §2.3 provides the announced parametrisation of  $\mathcal{G}$  in Proposition 2.29 and Theorem 2.31, and investigates in detail how it works in some particular cases: low nilpotence indices *etc.*

The case  $N^* = -N$  demands some additional theory — introducing counterparts of  $\partial$  and  $\bar{\partial}$ , of the Dolbeault lemma *etc.* We hope to publish this work later.

- Part 3 deals with  $\mathfrak{s}$  and  $\mathfrak{n}$  together: it uses parts 1 and 2 to show the theorem stated above, which gathers and generalises the most part of the preceding results. See Theorem 3.2. If  $g$  is hyperkähler *i.e.*  $H_{\mathfrak{s}} = \text{Sp}(p, q)$ , or is of a similar type, this is done by solving an exterior differential system exactly as is done by R. Bryant in [11], but in the framework of “ $\mathbb{R}[X]/(X^n)$ -differential calculus” introduced in part 2.

- Part 4, a lot shorter as well as Part 5, uses parts 1 and 2 to give a glimpse, through a simple example, on the case where  $\mathfrak{n}$  does not admit a unique generator.

- Part 5 consists of one single Theorem 5.1, studying the Ricci curvature of metrics admitting non trivial parallel endomorphisms.

**General setting and some general notation.** Here  $\mathcal{M}$  is a simply connected manifold of dimension  $d$  and  $g$  a Riemannian or pseudo-Riemannian metric on it, the holonomy representation of which does not stabilise any nondegenerate subspace, that is to say does not split in an orthogonal sum of subrepresentations. In particular,  $g$  does not split into a Riemannian product. We set  $H \subset \text{SO}^0(T_m\mathcal{M}, g|_m)$  the holonomy group of  $g$  at  $m$  and  $\mathfrak{h}$  its Lie algebra. As  $\mathcal{M}$  is supposed to be simply connected, we deal everywhere with  $\mathfrak{h}$ , forgetting  $H = \exp \mathfrak{h}$ . Let  $\mathfrak{e}$  be the algebra  $\text{End}(T_m\mathcal{M})^{\mathfrak{h}}$  of the parallel endomorphisms of  $g$  — to commute with  $\mathfrak{h}$  amounts to extend as a parallel field—; it is isomorphic to some subalgebra of  $M_d(\mathbb{R})^{\mathfrak{h}}$ . Notice that  $\mathfrak{e}$  is stable by  $g$ -adjunction, which we denote by  $\sigma : a \mapsto a^*$ . When lower case letters:  $x_i, y_i$  *etc.* stand for local coordinates, the corresponding upper case letters:  $X_i, Y_i$  *etc.* stand for the corresponding coordinate vector fields.

The matrix  $\text{diag}(I_p, -I_q) \in M_{p+q}(\mathbb{R})$  is denoted by  $I_{p,q}$ ,  $\begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix} \in M_{2p}(\mathbb{R})$  by  $J_p$  and  $\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} \in M_{2p}(\mathbb{R})$  by  $L_p$ . If  $V$  is a vector space of even dimension  $d$ , we recall that an  $L \in \text{End}(V)$  is called *paracomplex* if  $L^2 = \text{Id}$  with  $\dim \ker(L - \text{Id}) = \dim \ker(L + \text{Id}) = \frac{d}{2}$ .

Finally, take  $A \in \Gamma(\text{End}(T\mathcal{M}))$ , paracomplex or nilpotent. If it is integrable *i.e.* if its matrix is constant in well-chosen local coordinates, we call it a “paracomplex structure” or a “nilpotent structure”, like a complex structure, as opposed to an almost complex one.

**Thanks.** I thank M. Brion for a few crucial pieces of information and references in Algebra. I thank P. Baumann for his availability, and him, W. Bertram and V. S. Matveev for the references they indicated to me. I thank L. Bérard Bergery and S. Gallot for two indications, and M. Audin and P. Mounoud for their comments on the writing of certain parts of the manuscript.

## 1 The algebra $\mathfrak{e} = \text{End}(T\mathcal{M})^H$ and its semi-simple part $\mathfrak{s}$

### 1.1 The decomposition $\mathfrak{e} = \mathfrak{s} \ltimes \mathfrak{n}$ of $\mathfrak{e}$ in a semi-simple part and its radical

First we need to recall some facts and set some notation. All the results invoked are classical for finite dimensional associative algebras; we state them for a unitary real algebra  $A$ .

**1.1 Notation** If  $A$  is a subset of an algebra,  $A^* \subset A$  denotes here the subset of its invertible elements. If  $\sigma$  is an involutive anti morphism of  $A$ , then  $A^\pm = \{U \in A; \sigma(U) = \pm U\}$  denotes the subspace of its self adjoint or skew adjoint elements.

**1.2 Reminder** An algebra  $A$  is said to be *nilpotent* if  $A^k$ , the algebra spanned by the products of  $k$  elements of  $A$ , is  $\{0\}$  for some  $k$ . In particular, the elements of a nilpotent subalgebra of  $M_n(\mathbb{R})$  are simultaneously strictly upper triangular in some well-chosen basis.

**1.3 Definition** (See [12] §25 or [14]) The radical  $\text{Rad } A$  of  $A$  is the intersection of its maximal ideals. It is a nilpotent ideal. Equivalently, it is the sum of the nilpotent ideals of  $A$ . The algebra  $A$  is said to be *simple* if its only proper ideal is  $\{0\}$ , and *semi-simple* if its radical is  $\{0\}$  — so a simple algebra is semi-simple. Notice that  $A/\text{Rad}(A)$  is semi-simple.

The decomposition  $\mathfrak{e} = \mathfrak{s} \ltimes \mathfrak{n}$  is provided by the following classical result. The last assertion is a refinement due to Taft [23, 24]. I thank P. Baumann for this reference.

**1.4 Theorem [Wedderburn – Malčev]** (see [12] §72) Let  $A$  be a finite dimensional  $\mathbb{R}$ -algebra. Then there exists a semi-simple algebra  $A_S$  in  $A$  such that  $A = A_S \ltimes \text{Rad}(A)$ . If moreover  $A$  is endowed with an involutive anti-morphism  $\sigma$ , then  $A_S$  may be chosen  $\sigma$ -stable.

**1.5 Notation** We set  $\mathfrak{n} = \text{Rad } \mathfrak{e}$ . Being the unique maximal nilpotent ideal of  $\mathfrak{e}$ ,  $\mathfrak{n}$  is self adjoint that is to say stable by  $g$ -adjunction. We take  $\mathfrak{s} \simeq \mathfrak{e}/\mathfrak{n}$  some self adjoint semi-simple subalgebra of  $\mathfrak{e}$  as provided by Theorem 1.4.

### 1.2 Some natural objects associated with a reducible holonomy representation; a "quasi-commutation" property in $\mathfrak{e}$

**1.6 Remark/Notation** We denote by  $E_0 = \bigcap_{W \in \mathfrak{h}} \ker W$  the (possibly trivial) maximal subspace of  $T_m\mathcal{M}$  on which the holonomy group  $H$  acts trivially. As  $T_m\mathcal{M}$  is  $H$ -orthogonally indecomposable,  $E_0$  is totally isotropic. We set  $\mathfrak{n}_0 = \{N \in \mathfrak{e}; \text{Im } N \subset E_0\}$ ; as the actions of  $H$  and  $\mathfrak{e}$  on  $T_m\mathcal{M}$  commute,  $\mathfrak{n}_0$  is an ideal of  $\mathfrak{e}$ , moreover adjunction-stable. So, for any  $x, y \in T_m\mathcal{M}$ , and any  $N, N' \in \mathfrak{n}_0$ ,  $g(N'Nx, y) = g(Nx, N'^*y) \in g(E_0, E_0) = \{0\}$ , so  $N'N = 0$  *i.e.*  $\mathfrak{n}_0^2 = \{0\}$ .

**1.7 Remark/Notation** The algebra  $\mathfrak{e}$  is naturally endowed with the bilinear symmetric form  $\langle U, V \rangle = \frac{1}{d} \text{tr}(U^*V)$ . By Reminder 1.2,  $\mathfrak{n} \subset \ker(\langle \cdot, \cdot \rangle)$ . If moreover  $\mathfrak{e}$  admits some self adjoint complex structure  $\underline{J}$ , and denoting by  $\mathfrak{e}_{\underline{J}}$  the  $\underline{J}$ -complex algebra  $\{U \in \mathfrak{e}; U\underline{J} = \underline{J}U\}$ , then  $\mathfrak{e}_{\underline{J}}$  is endowed with the complex form  $\langle U, V \rangle_{\underline{J}} = \frac{1}{d}(\text{tr}(U^*V) - i \text{tr}(U^*\underline{J}V))$ .

The following proposition is the key of most steps of the classification of the possible  $\mathfrak{s}$ . As it is also worth to be noticed by itself, we state it apart, here.

**1.8 Proposition** *Let  $U, V$  be in  $\mathfrak{e}$  and  $m$  be any point of  $\mathcal{M}$ . Then, if  $U$  is self adjoint, then for any  $x, y \in T_m\mathcal{M}$ ,  $R(x, y) \circ (UV - VU) = 0$ . Consequently,  $UV - VU \in \mathfrak{n}_0$ .*

**1.9 Remark** In particular, in case  $E_0 = \cap_{W \in \mathfrak{h}} \ker W = \{0\}$ , all self adjoint elements of  $\mathfrak{e}$  are central in  $\mathfrak{e}$ .

Proposition 1.8 rests on the following remark.

**1.10 Reminder/Remark** We will need the following remark. The Bianchi identity implies that, at any point  $m \in \mathcal{M}$  and for any  $x, y, z, t \in T_m\mathcal{M}$ ,  $g(R(x, y)z, t) = g(R(z, t)x, y)$ . This relation holds also for any bilinear form  $g'$ , parallel with respect to the Levi Civita connection of  $g$ , be  $g'$  nondegenerate or not. The proof does not need nondegeneracy, see *e.g.* Lemma 9.3 in [19]. Consequently, if  $U$  is a parallel self adjoint endomorphism, then  $R(Ux, y)z = R(x, Uy)z = R(x, y)Uz$ . The first equality is classical. For the second one, take  $t$  any fourth vector and denote by  $g_U$  the bilinear form  $= g(\cdot, U \cdot)$ , which is parallel, as  $U$  is, and symmetric, as  $U$  is  $g$ -self adjoint. Then:

$$\begin{aligned} g(R(x, Uy)z, t) &= g(R(z, t)x, Uy) && \text{applying the relation to } g, \\ &= g_U(R(z, t)x, y) \\ &= g_U(R(x, y)z, t) && \text{applying the relation to } g_U, \\ &= g(R(x, y)z, Ut) \\ &= g(R(x, y)Uz, t) && \text{as } U^* = U, \text{ being parallel, commutes with } R(x, y). \end{aligned}$$

**Proof of Proposition 1.8.** Take  $U, V \in \mathfrak{e}$  with  $U^* = U$  and  $x, y, z, t \in T_m\mathcal{M}$ . The bilinear form  $g_U := g(\cdot, U \cdot)$  is parallel, as  $U$  is.

$$\begin{aligned} g(R(x, y)z, VUt) &= g(R(x, y)V^*z, Ut) && \text{as, } V^*, \text{ parallel, commutes with } R(x, y), \\ &= g(R(x, Uy)V^*z, t) && \text{by Remark 1.10, applied to } U, \\ &= g(R(x, Uy)z, Vt) && \text{as, } V^* \text{ commutes with } R(x, y), \\ &= g(R(x, y)z, UVt) && \text{again by Remark 1.10, so the result. } \quad \square \end{aligned}$$

### 1.3 The eight possible forms of $\mathfrak{s}$

Theorem 1.11 gives them. Each type is well-known, except possibly  $(\mathbf{3}^{\mathbb{C}})$ : so the theorem closes the list, may the action of  $H$  be totally reducible or not. The proof rests on the classical Wedderburn-Artin and Skolem Noether theorems, and then is elementary. Remark 1.15 below gives the generic holonomy group corresponding to each case of the theorem.

**1.11 Theorem** *The algebra  $\mathfrak{s}$  is of one of the following types, where  $\underline{J}$ ,  $J$ , and  $L$  denote respectively self adjoint complex structures and skew adjoint complex and paracomplex structures.*

(1) **generic**,  $\mathfrak{s} = \text{span}(\text{Id})$ .

(1<sup>ℂ</sup>) “**complex Riemannian**”,  $\mathfrak{s} = \text{span}(\text{Id}, \underline{J})$ . Here  $d \geq 4$  is even,  $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$ ,  $(\mathcal{M}, J, g(\cdot, \cdot) - \text{ig}(\cdot, \underline{J}\cdot))$  is complex Riemannian for a unique complex structure in  $\mathfrak{s}$ , up to sign.

(2) (pseudo-)Kähler,  $\mathfrak{s} = \text{span}(\text{Id}, J)$ . Here  $d$  is even and  $(\mathcal{M}, J, g)$  is (pseudo-)Kähler, for a unique complex structure in  $\mathfrak{s}$ , up to sign.

(2') parakähler,  $\mathfrak{s} = \text{span}(\text{Id}, L)$ . Here  $d$  is even,  $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$ ,  $(\mathcal{M}, L, g)$  is parakähler, for a unique paracomplex structure in  $\mathfrak{s}$ , up to sign.

(2<sup>ℂ</sup>) “**complex Kähler**”,  $\mathfrak{s} = \text{span}(\text{Id}, \underline{J}, L, J)$ . Here  $d \in 4\mathbb{N}^*$ ,  $\text{sign}(g) = (\frac{n}{2}, \frac{n}{2})$  and  $(\mathcal{M}, \underline{J}, J, L, g)$  is at once complex Riemannian, pseudo-Kähler, and parakähler, on a unique way in  $\mathfrak{s}$ , up to sign of each structure.

(3) (pseudo-)hyperkähler,  $\mathfrak{s} = \text{span}(\text{Id}, J_1, J_2, J_3)$ . Here  $d \in 4\mathbb{N}^*$ ,  $(\mathcal{M}, J_1, J_2, g)$  is (pseudo-)hyperkähler, the set of Kähler structures in  $\mathfrak{s}$  being a 2-dimensional submanifold.

(3') “**para-hyperkähler**”,  $\mathfrak{s} = \text{span}(\text{Id}, J, L_1, L_2)$ . Here  $d \in 4\mathbb{N}^*$ ,  $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$  and  $(\mathcal{M}, J, L_1, g)$  is at once pseudo-Kähler and parakähler, the set of complex and of paracomplex structures in  $\mathfrak{s}$  being each a 2-dimensional submanifold.

(3<sup>ℂ</sup>) “**complex hyperkähler**”,  $\mathfrak{s} = \text{span}(\text{Id}, \underline{J}, J, L_1, L_2, \underline{J}J, \underline{J}L_1, \underline{J}L_2)$ . Here  $d \in 8\mathbb{N}^*$ ,  $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$  and  $(\mathcal{M}, g)$  is at once complex Riemannian (on a unique way up to sign in  $\mathfrak{s}$ ), and pseudo-Kähler and parakähler. The sets of pseudo- or parakähler structures are 2-dimensional  $\underline{J}$ -complex submanifolds of  $\mathfrak{s}$ .

Each case is precisely described in Tables 1 p. 8 and 2 p. 9, which are part of the theorem.

Each type is produced by a non-empty set of germs of metrics. On a dense open subset of them, for the  $C^2$  topology, the holonomy group of the metric is the commutant  $\text{SO}^0(g)^{\mathfrak{s}}$  of  $\mathfrak{s}$  in  $\text{SO}^0(g)$ .

**1.12 Remark** The fact that the set of germs of metrics in each case is non empty is well-known, except perhaps for types (3') and (3<sup>ℂ</sup>). In all cases, §1.4 gives their parametrisation.

**1.13 Notation** If  $G$  is a subgroup of  $\text{GL}_d(\mathbb{K})$ , we denote here by  $\mathbf{V}$  its standard representation in  $\mathbb{K}^d$ . We denote then by  $\mathbf{V}^* : g \mapsto (\lambda \mapsto \lambda \circ g^{-1})$  its contragredient representation in  $(\mathbb{K}^d)^*$  and, if  $\mathbb{K} = \mathbb{C}$ , by  $\overline{\mathbf{V}}^*$  the complex conjugate of it.

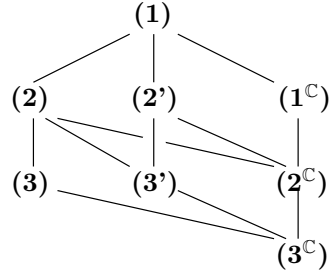
**1.14 Remark** In cases (2'), (2<sup>ℂ</sup>), (3') and (3<sup>ℂ</sup>), the existence of a paracomplex structure  $L$  splits  $\text{T}\mathcal{M} = \ker(L - \text{Id}) \oplus \ker(L + \text{Id}) = V \oplus V'$  into a sum of two totally isotropic factors, and the morphism  $\flat$  given by the metric identifies  $V'$  with  $V^*$ . Then,  $H$  is isomorphic to a subgroup  $[H]$  of  $\text{GL}_{d/2}(\mathbb{K})$ , the holonomy representation being  $\mathbf{V} \oplus \mathbf{V}^*$ , if  $\mathbb{K} = \mathbb{R}$ , or  $\mathbf{V} \oplus \overline{\mathbf{V}}^*$ , if  $\mathbb{K} = \mathbb{C}$ , on  $\ker(L - \text{Id}) \oplus \ker(L + \text{Id})$ . Matricially,  $H := \left\{ \begin{pmatrix} U & 0 \\ 0 & {}_t U^{-1} \end{pmatrix}, U \in [H] \right\}$ ; so if  $\mathbb{K} = \mathbb{R}$ ,  $H \subset \text{SO}^0(\frac{d}{2}, \frac{d}{2})$  and if  $\mathbb{K} = \mathbb{C}$ ,  $H \subset \text{U}(\frac{d}{2}, \frac{d}{2})$ .

**1.15 Remark** For  $\mathfrak{s}$  of each type, we sum up here: the possible signature(s) of  $g$ , the group in which  $H$  (possibly identified with  $[H]$ , see Rem. 1.14) is included, and to which it is generically equal — see §1.4 for a proof —, and the representation of  $H$  or  $[H]$ . Notice that each time, this group is also the commutant of  $\mathfrak{s}$  in  $\text{SO}^0(g)$ . See Notation 1.13 for  $\mathbf{V}$ .

(1)	(1 <sup>ℂ</sup> )	(2)	(2')	(2 <sup>ℂ</sup> )	(3)	(3')	(3 <sup>ℂ</sup> )
$(p, q)$	$(p, p)$	$(2p, 2q)$	$(p, p)$	$(2p, 2p)$	$(4p, 4q)$	$(2p, 2p)$	$(4p, 4p)$
$\text{SO}^0(p, q)$	$\text{SO}(p, \mathbb{C})$	$\text{U}(p, q)$	$\text{GL}^0(p, \mathbb{R})$	$\text{GL}(p, \mathbb{C})$	$\text{Sp}(p, q)$	$\text{Sp}(2p, \mathbb{R})$	$\text{Sp}(2p, \mathbb{C})$
$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V}$	$\mathbf{V} \oplus \mathbf{V}^*$	$\mathbf{V} \oplus \overline{\mathbf{V}}^*$	$\mathbf{V}$	$\mathbf{V} \oplus \mathbf{V}^*$	$\mathbf{V} \oplus \overline{\mathbf{V}}^*$

**1.16 Remark** In Theorem 1.11, the new cases with respect to the Riemannian framework occur only for metrics  $g$  of signature  $(\frac{d}{2}, \frac{d}{2})$ .

**1.17 Remark [Justification of the labels in Theorem 1.11]** The generic holonomy groups corresponding to  $\mathfrak{s}$  of types  $(1^{\mathbb{C}})$ ,  $(2^{\mathbb{C}})$  and  $(3^{\mathbb{C}})$  are complexification of those corresponding to  $\mathfrak{s}$  of respective types  $(1)$ ,  $((2)$  or  $(2')$ ), and  $((3)$  or  $(3'))$ . Besides, if you consider the different types in a comprehensive sense, type  $(2)$  *e.g.* meaning only “ $H \subset U(p, q)$ ”, and so on, you obtain the following inclusion diagram:



where the strokes denote the inclusion of the set of metrics below into the one above.

This justifies our notation, included the slight formal dissymmetry between  $((2), (3))$  and  $((2'), (3'))$ . Another point of view is the following. Suppose that  $g$  is a real analytic germ of metric at  $m$ . Then  $\mathfrak{h}$  is generated by  $\{D^k R(u_1, \dots, u_{k+2}), (u_i)_{i=1}^{k+2} \in T_m \mathcal{M}\}$ , the curvature tensors at  $m$  and their covariant derivatives at all orders. So the complexification  $g^{\mathbb{C}}$  of the germ  $g$  has  $H \otimes \mathbb{C}$  as holonomy group. Thus here, if  $g$  is “of type  $(1)$ ”, respectively  $((2)$  or  $(2')$ ), or  $((3)$  or  $(3'))$ , its complexification is “of type  $(1^{\mathbb{C}})$ ”, respectively  $(2^{\mathbb{C}})$  and  $(3^{\mathbb{C}})$ .

**1.18 Lemma** Take  $U \in \mathfrak{e}$  and  $N \in \mathfrak{n}$ ,  $\mu$  the minimal polynomial of  $U$  and  $\mu'$  that of  $U + N$ . Then any irreducible factor of  $\mu$  is also in  $\mu'$ , and vice versa.

**Proof.**  $\mu(U + N) = \mu(U) + NV = NV$  with  $V$  some polynomial in  $U$  and  $N$ . As  $\mathfrak{n}$  is an ideal,  $NV \in \mathfrak{n}$  and by Proposition 1.2,  $NV$  is nilpotent. So for some  $k \in \mathbb{N}$ ,  $(\mu^k)(U + N) = 0$  *i.e.*  $\mu' | \mu^k$ . Symmetrically,  $\exists l \in \mathbb{N}^* : \mu | \mu'^l$ , so the result.  $\square$

**Proof of Theorem 1.11.** We denote  $T_m \mathcal{M}$  by  $E$  in this proof. We first state the announced classical results in associative algebra.

**1.19 Theorem [Wedderburn – Artin]** (see [14] §3, p. 40). Let  $A$  be a finite dimensional semi-simple  $\mathbb{R}$ -algebra. Then  $A$  is isomorphic to a direct sum of matrix algebras:  $A \simeq \bigoplus_{i=1}^k M_{d_i}(\mathbb{K}_i)$ , with for each  $i$ ,  $d_i \in \mathbb{N}^*$  and  $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ .

**1.20 Theorem [corollary of a theorem of Skolem – Noether]** (see [9] §10, no. 1). Let  $A$  be a finite dimensional semi-simple  $\mathbb{R}$ -algebra and  $\theta$  an automorphism of  $A$  acting trivially on the center of  $A$ . Then  $\theta$  is interior.

As  $g$  is orthogonally indecomposable, then if it is flat,  $\dim \mathcal{M} = 1$  and  $\mathfrak{s} = \mathfrak{e} = \mathbb{R} \text{Id}$  is of type  $(1)$ . We now suppose that  $g$  is not flat. The elimination of only one possible form for  $\mathfrak{s}$  will follow, through Proposition 1.8, from the fact that  $\mathfrak{h}$  is a holonomy algebra *i.e.* from the Bianchi identity satisfied by the curvature tensor. All the other eliminations follow from the orthogonal indecomposability of the action of  $\mathfrak{h}$ , through the following claim.

*Claim 1.* If  $p \in \mathfrak{e}$  is self adjoint, its minimal polynomial  $\mu_p(X)$  is of the form  $Q^\alpha$  with  $Q$  irreducible. For example, if  $p$  is not invertible, it is nilpotent.

case	$\mathfrak{s}^+ \subset \mathfrak{s}$	$\mathfrak{s}$ given as a unitary $\mathbb{R}$ -algebra generated by $\langle \cdot, \cdot \rangle$ -orthogonal complex and para-complex structures	corresponding generators of $A$	$g$ -adjunction $\sigma$ in $\mathfrak{s}$ corresponds in $A$ to
(1)	$\mathbb{R} \subset \mathbb{R},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{1,0}$	$\langle \cdot \rangle$	$\emptyset$	Id
(1 <sup>C</sup> )	$\mathbb{C} \subset \mathbb{C},$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{1,1}$	$\langle \underline{J} \rangle$	$\{i\}$	Id $_{\mathbb{C}}$
(2)	$\mathbb{R} \subset \mathbb{C},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{2,0}$	$\langle J \rangle$	$\{i\}$	$z \mapsto \bar{z}$
(2')	$\mathbb{R} \cdot (1,1) \subset \mathbb{R} \oplus \mathbb{R},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{1,1}$	$\langle L \rangle$	$\{(1, -1)\}$	$(a, b) \mapsto (b, a)$
(2 <sup>C</sup> )	$\mathbb{C} \cdot (1,1) \subset \mathbb{C} \oplus \mathbb{C},$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{2,2}$	$\langle \underline{J}, L, J \mid$ $\underline{J} \in Z(\mathfrak{s}),$ $LJ = JL = \underline{J} \rangle$	$\{(i, i), (1, -1),$ $(i, -i)\}$	$(a, b) \mapsto (b, a)$
(3)	$\mathbb{R} \subset \mathbb{H},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{4,0}$	$\langle J_1, J_2, J_3 \mid$ $J_{[i]}J_{[i+1]} = J_{[i+2]},$ $J_i J_{i'} = -J_{i'} J_i \rangle$	the canonical $\{i, j, k\} \subset \mathbb{H}$	$z \mapsto \bar{z}$ (quaternionic conjugation)
(3')	$\mathbb{R} \cdot I_2 \subset M_2(\mathbb{R}),$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{2,2}$	$\langle L_1, L_2, J \mid$ $L_1 L_2 = -L_2 L_1 = -J,$ $LJ_2 = -JL_2 = L_1,$ $JL_1 = -L_1 J = L_2 \rangle$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \right.$ $\left. \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ i.e. transpose of the comatrix
(3 <sup>C</sup> )	$\mathbb{C} \cdot I_2 \subset M_2(\mathbb{C}),$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{4,4}$	$\langle \underline{J}, L_1, L_2, J \mid$ $\underline{J} \in Z(\mathfrak{s}),$ $L_1 L_2 = -L_2 L_1 = -J,$ $L_2 J = -JL_2 = L_1,$ $JL_1 = -L_1 J = L_2 \rangle$	$\{iI_2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ i.e. transpose of the comatrix
(2 <sup>C</sup> ): $\mathfrak{s}$ given as a $\underline{J}$ -complex algebra		$\langle L \rangle$	$\{(1, -1)\}$	same as above
(3 <sup>C</sup> ): other presentation of $\mathfrak{s}$ , given as a $\underline{J}$ -complex algebra		$\langle L_1, L_2, L_3 \mid$ $iL_{[i]}L_{[i+1]} = L_{[i+2]},$ $L_{[i]}L_{[i']} = -L_{[i']}L_{[i]} \rangle$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \right.$ $\left. \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}$	same as above

Table 1: *Theorem 1.11 summed up in a table.* All letters  $J$  denote complex structures, and  $L$  paracomplex ones. All are  $g$ -skew adjoint, except the  $g$ -self adjoint underlined  $\underline{J}$ . Bracketed indices  $[i]$  denote indices modulo 3. Column 2 gives the inclusion  $\mathfrak{s}^+ \subset \mathfrak{s}$ , first as an inclusion of algebras, then as inclusion of (pseudo-)euclidian spaces for  $\langle \cdot, \cdot \rangle$ ;  $\mathbb{R}^{a,b}$  means  $(R^{a+b}, \langle \cdot, \cdot \rangle)$  with  $\text{sign}(\langle \cdot, \cdot \rangle) = (a, b)$ . All  $\mathfrak{s}$  and  $\mathfrak{s}^+$  are  $\langle \cdot, \cdot \rangle$ -nondegenerate.

*Proof.* The minimal polynomial reads  $\mu_p(X) = \prod_{i=1}^N Q_i^{\alpha_i}$  with irreducible  $Q_i$ 's. As  $p$  is self adjoint, the direct sum  $E = \bigoplus_{i=1}^N \ker Q_i^{\alpha_i}(p)$  is  $g$ -orthogonal. As  $p \in \text{End}(E)^{\mathfrak{h}}$ , each  $\ker Q_i^{\alpha_i}(p)$  is  $\mathfrak{h}$ -stable. Now  $E$  is indecomposable, so  $N = 1$  and the claim.

By 1.4,  $\mathfrak{e} = \mathfrak{s} \ltimes \mathfrak{n}$  where  $\mathfrak{n} = \text{Rad}(\mathfrak{e})$  and  $\mathfrak{s}$  is a semi-simple, adjunction-stable subalgebra of  $\mathfrak{e}$ . As  $\mathfrak{n}$  is the intersection of the maximal ideals of  $\mathfrak{e}$  and as the adjunction is an anti-morphism,  $\mathfrak{n}$  is necessarily also adjunction-stable. By 1.19, we find an isomorphism  $\varphi : \mathfrak{s} \xrightarrow{\sim} A$  with  $A = \bigoplus_{i=1}^k I_i$  and  $I_i = M_{\delta_i}(\mathbb{K}_i)$  as in 1.19. By a slight abuse, we also denote by  $\sigma$  the conjugate action  $\varphi \circ \sigma \circ \varphi^{-1}$  of  $\sigma$  on  $A$ .

*Claim 2.*  $k \leq 2$ . If  $k = 2$ , then  $\sigma(I_1) = I_2$ . We then denote the  $\delta_i$  by  $\delta$  and the  $\mathbb{K}_i$  by  $\mathbb{K}$ .



case	$\dim \mathcal{M},$ $(\cdot, \cdot) = \text{sign}(g)$	matrices in a well-chosen basis:	“Complex Riemannian” structures	Kähler structures	para Kähler structures
(1)	any, any	$\text{Mat}(g) = I_{p,q}$	$\emptyset$	$\emptyset$	$\emptyset$
(1 <sup>c</sup> )	$d = 2p,$ $(p, p)$	$\text{Mat}(g) = I_{p,p}$ $\text{Mat}(\underline{J}) = J_p$	$\{\pm \underline{J}\}$	$\emptyset$	$\emptyset$
(2)	$d \in 2\mathbb{N}^*,$ $(2p, 2q),$ $p + q = \frac{d}{2}$	$\text{Mat}(g) =$ $\text{diag}(I_{p,q}, I_{p,q})$ $\text{Mat}(J) = J_{d/2}$	$\emptyset$	$\{\pm J\}$	$\emptyset$
(2')	$d = 2p,$ $(p, p)$	$\text{Mat}(g) = I_{p,p}$ $\text{Mat}(L) = L_p$ [1]	$\emptyset$	$\emptyset$	$\{\pm L\}$
(2 <sup>c</sup> )	$d = 4p,$ $(2p, 2p)$	$\text{Mat}(g) = L_{2p}$ $\text{Mat}(L) = I_{2p, 2p}$ $\text{Mat}(\underline{J}) =$ $\text{diag}(J_p, -J_p)$	$\{\pm \underline{J}\}$	$\{\pm J\}$	$\{\pm L\}$
(3)	$d \in 4\mathbb{N}^*,$ $(4p, 4q),$ $p + q = \frac{d}{4}$	$\text{Mat}(g) =$ $\text{diag}(I_{p,q}, I_{p,q}, I_{p,q}, I_{p,q})$ $\text{Mat}(J_1) =$ $\text{diag}(-J_{d/4}, J_{d/4})$ $\text{Mat}(J_2) = J_{d/2}$ $\text{Mat}(J_3) = \begin{pmatrix} 0 & J_{d/4} \\ J_{d/4} & 0 \end{pmatrix}$	$\emptyset$	the 2-sphere $\{\alpha J_1 + \beta J_2 + \gamma J_3;$ $\alpha^2 + \beta^2 + \gamma^2 = 1\}$ $= \{U; \langle U, U \rangle = 1\}$	$\emptyset$
(3')	$d = 4p$ $(2p, 2p)$	$\text{Mat}(g) = I_{2p, 2p}$ $\text{Mat}(L_1) = L_{2p}$ $\text{Mat}(L_2) = \begin{pmatrix} 0 & -J_p \\ J_p & 0 \end{pmatrix}$ $\text{Mat}(J) =$ $\text{diag}(-J_p, J_p)$ [2]	$\emptyset$	the two-sheet hyperboloid $\{\alpha L_1 + \beta L_2 + \gamma J;$ $\alpha^2 + \beta^2 - \gamma^2 = -1\}$ $= \{U; \langle U, U \rangle = 1\}$	the one-sheet hyperboloid $\{\alpha L_1 + \beta L_2 + \gamma J;$ $\alpha^2 + \beta^2 - \gamma^2 = 1\}$ $= \{U; \langle U, U \rangle = -1\}$
(3 <sup>c</sup> )	$d = 8p$ $(4p, 4p)$	$\text{Mat}(g) =$ $\text{diag}(I_{2p, 2p}, -I_{2p, 2p})$ $\text{Mat}(\underline{J}) = \text{diag}(J_{2p}, J_{2p})$ $\text{Mat}(L_1) = L_{4p}$ $\text{Mat}(L_2) =$ $\begin{pmatrix} 0 & \text{diag}(J_p, J_p) \\ -\text{diag}(J_p, J_p) & 0 \end{pmatrix}$ $\text{Mat}(J) =$ $\text{diag}(J_p, J_p, -J_p, -J_p)$	$\{\pm \underline{J}\}$	seeing $\mathfrak{s}$ as $\underline{J}$ -complex, the proper quadric with centre $\{\alpha L_1 + \beta L_2 + \gamma J;$ $\alpha = \alpha' + \alpha'' \underline{J}$ etc. $\alpha^2 + \beta^2 - \gamma^2 = -1\}$ $= \{U; \langle U, U \rangle_{\underline{J}} = 1\}$	seeing $\mathfrak{s}$ as $\underline{J}$ -complex, the proper quadric with centre $\{\alpha L_1 + \beta L_2 + \gamma J;$ $\alpha = \alpha' + \alpha'' \underline{J}$ etc. $\alpha^2 + \beta^2 - \gamma^2 = 1\}$ $= \{U; \langle U, U \rangle_{\underline{J}} = -1\}$
[1] or e.g.: $\text{Mat}(g) = L_n, \text{Mat}(L) = I_{n,n}.$					
[2] or e.g.: $\text{Mat}(g) = L_n, \text{Mat}(L_1) = I_{2n, 2n}, \text{Mat}(L_2)$ unchanged, $\text{Mat}(J) = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix}.$					
Cases (3), (3'), and (3 <sup>c</sup> ) imply that the metric is Ricci-flat.					

Table 2: *Theorem 1.11 given in a matricial form, together with the different complex Riemannian and (para) Kähler structures appearing in each case.*

*Proof.* Let us denote by 1 the unit matrix of any factor of  $A$ . Notice that  $\sigma$ , as any (anti) automorphism of  $A$ , acts on the factors  $I_i$  of  $A$ , permuting them. Take  $p = (1, 0, \dots, 0) \in A$ . As  $p^2 = p$ ,  $\varphi^{-1}(p)$  is a (non zero) projection, so by Claim 1, either  $\varphi^{-1}(p) = 1_{\mathfrak{e}}$  and thus  $k = 1$ , or  $\sigma(p) \neq p$ . In the latter case,  $\sigma(I_1) \neq I_1$ . Take  $p' = p + \sigma(p)$ . It is selfadjoint by construction, and  $p'^2 = p^2 + \sigma(p)^2 = p + \sigma(p^2) = p'$  so it is a (non zero) projection. By Claim 1,  $\varphi^{-1}(p') = 1_{\mathfrak{e}}$  so  $A = I_1 \oplus \sigma(I_1)$  and then  $k = 2$ .

*Claim 3.* If  $k = 2$ , then  $\delta = 1$  and  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ .

*Proof.* Suppose  $k = 2$ . Take  $p = (\text{diag}(1, 0, \dots, 0), 0) \in A$  and  $p' = p + \sigma(p)$ . By the same reasoning as above,  $\varphi^{-1}(p')$  is a non zero self adjoint projection so  $p' = 1_A$  by Claim 1. As the  $I_1$ -component of  $\sigma(p)$  is zero, then in fact  $p = (1, 0)$  and  $\sigma(p) = (0, 1)$ ; in particular  $\text{diag}(1, 0, \dots, 0) = 1_{I_1}$  i.e.  $\delta = 1$ . To eliminate the case  $\mathbb{K} = \mathbb{H}$ , we use Proposition 1.8. Suppose that  $\mathbb{K} = \mathbb{H}$ , denote by  $i$  and  $j$  two of the three canonical roots of  $-1$  in  $\mathbb{H}$ , take  $m \in \mathcal{M}$  and  $x, y \in T_m \mathcal{M}$  and take  $I = \varphi^{-1}(i, 0)$  and  $J = \varphi^{-1}(j, 0)$  in  $\mathfrak{e} = \varphi^{-1}(\mathbb{H} \oplus \mathbb{H})$ . Notice that the  $I_1$  component of  $\sigma((i, 0))$  is zero, so  $I^*J = 0$ , similarly  $IJ^* = 0$ . By construction,  $I + I^*$  is self adjoint, so:

$$\begin{aligned} R(x, y).(I + I^*)(J + J^*) &= R(x, y).(J + J^*)(I + I^*) && \text{by Proposition 1.8,} \\ &= R(x, y).(JI + J^*I^*) \\ &= -R(x, y).(IJ + I^*J^*) && \text{as in } \mathbb{H}, ji = -ij, \\ &= -R(x, y).(I + I^*)(J + J^*). \end{aligned}$$

So  $R(x, y).(I + I^*)(J + J^*) = 0$ . Now  $(I + I^*)(J + J^*) = IJ + (IJ)^* = \varphi^{-1}((ij, 0) + \sigma((ij, 0)))$  is invertible, so for any  $m \in \mathcal{M}$  and any  $x, y \in T_m \mathcal{M}$ ,  $R(x, y) = 0$  i.e.  $(\mathcal{M}, g)$  is flat, in contradiction with  $\mathfrak{s} \simeq \mathbb{H} \oplus \mathbb{H}$ . So  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ .

Let us suppose  $k = 1$  and finish the proof. Denote by  $\tau$  the transposition  $u \mapsto {}^t u$  in  $A \simeq M_\delta(\mathbb{K})$ , and by  $\bar{\tau}$  its composition  $u \mapsto {}^t \bar{u}$  with the conjugation, in case  $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ . Then for  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , respectively for  $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ ,  $\tau$ , respectively  $\bar{\tau}$ , is an anti-morphism (of  $\mathbb{R}$ -algebra) of  $A$ . So either  $\tau \circ \sigma$  or  $\bar{\tau} \circ \sigma$  is an automorphism of  $A$  and, for  $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$ , it acts trivially on the center  $Z(A)$  as  $Z(A) = \mathbb{K}.I_\delta$ . If  $\mathbb{K} = \mathbb{C}$ , either  $\sigma \circ \tau$  or  $\sigma \circ \bar{\tau}$  acts trivially on the center  $Z(A) = \mathbb{C}.I_\delta$ . Applying Theorem 1.20, we get a  $v \in A$  such that  $\sigma : u \mapsto v {}^t \tilde{u} v^{-1}$  with  $\tilde{u} = u$  if  $\mathbb{K} = \mathbb{R}$ ,  $\tilde{u} = \bar{u}$  if  $\mathbb{K} = \mathbb{H}$  and  $\tilde{u} = u$  or  $\tilde{u} = \bar{u}$  if  $\mathbb{K} = \mathbb{C}$ . Notice that, as  $\sigma^2 = \text{Id}_{\mathfrak{e}}$ ,  $v {}^t \tilde{v} v^{-1} \in Z(A)$  i.e.  ${}^t \tilde{v} = \lambda v$  with  $\lambda \in \mathbb{R}$  if  $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$  and  $\lambda \in \mathbb{C}$  if  $\mathbb{K} = \mathbb{C}$ . Applying  $\tilde{\tau}$  on both sides, we get that  $\lambda = \pm 1$  (in the case  $\mathbb{K} = \mathbb{C}$  and  $\tilde{u} = \bar{u}$ , we get only  $|\lambda| = 1$ , but replacing  $v$  by an adequate element of  $\mathbb{C}.v$  achieves even  $\lambda = 1$ ).

If we replace  $\varphi$  by  $c_w \circ \varphi$  with  $c_w : u \mapsto w^{-1} u w$ , then  $v$  is replaced by  $w v {}^t \tilde{w}$  i.e.  $v$  undergoes a basis change like the matrix of a bilinear or  $\sim$ -sesquilinear form. So using a suitable  $c_w$ , and recalling that  ${}^t \tilde{v} = \lambda v$  with  $\lambda = \pm 1$ , we may suppose:

- in case  $\lambda = 1$ , that  $v = \text{diag}(I_{\delta'}, -I_{\delta''})$  with  $\delta' + \delta'' = \delta$  if  $\mathbb{K} = \mathbb{R}$  or  $(\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\} \text{ and } \tilde{u} = \bar{u})$ , and that  $v = I_\delta$  if  $(\mathbb{K} = \mathbb{C} \text{ and } \tilde{u} = u)$ ,
- in case  $\lambda = -1$ , that  $\delta$  is even and  $v = \begin{pmatrix} 0 & -I_{\delta/2} \\ I_{\delta/2} & 0 \end{pmatrix}$  if  $(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \text{ and } \tilde{u} = u)$ , and that  $v = I_{\delta}.i$  if  $\mathbb{K} = \mathbb{H}$ .

Now all cases where  $v$  is diagonal imply  $\delta = 1$ . Indeed, if  $v = \text{diag}(I_{\delta'}, -I_{\delta''})$  or  $v = I_\delta$ , set  $p = \text{diag}(1, 0, \dots, 0)$ , and if  $\mathbb{K} = \mathbb{H}$  and  $v = I_{\delta}.i$ , set  $p = \text{diag}(j, 0, \dots, 0)$ . Then  $p$  is self adjoint, non nilpotent, so  $p = 1_A$  or  $p = 1_A.j$  by Claim 1 i.e.  $\delta = 1$ . So if  $\delta \geq 2$ , then  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $\lambda = -1$ ,  $\tilde{u} = u$ ,  $\delta$  is even and  $v = \begin{pmatrix} 0 & -I_{\delta/2} \\ I_{\delta/2} & 0 \end{pmatrix}$ . Setting  $p' = \text{diag}(1, 0, \dots, 0) \in M_{\delta/2}(\mathbb{K})$  we get  $p = \text{diag}(p', p')$  a self adjoint non nilpotent element of  $A$ , so  $p$  is invertible by Claim 1 i.e.  $\delta' = 1$  i.e.  $\delta = 2$ . So the only allowed cases are:

- if  $k = 1$  and  $\delta = 1$ ,  $\mathbb{K} = \mathbb{R}$  and  $\sigma : u \mapsto {}^t u = u$ , or  $\mathbb{K} = \mathbb{C}$  and  $\sigma : u \mapsto {}^t u = u$ , or  $\mathbb{K} = \mathbb{C}$  and  $\sigma : u \mapsto {}^t \bar{u} = \bar{u}$ , or  $\mathbb{K} = \mathbb{H}$  and  $\sigma : u \mapsto {}^t \bar{u} = \bar{u}$ ,
- if  $k = 1$  and  $\delta = 2$ ,  $(\mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C})$  and  $\sigma : u \mapsto v {}^t u v^{-1}$  with  $v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  i.e.  $\sigma$  is as described in Table 1,

– if  $k = 2$  and  $\delta = 1$  i.e.  $A = I_1 \oplus I_2$  with  $I_1 \simeq I_2 \simeq \mathbb{K}$ , ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ ) and  $\sigma$  permutes  $I_1$  and  $I_2$ . Composing possibly  $\varphi$  with a suitable automorphism of  $A$ , we get simply  $\sigma : (a, b) \mapsto (b, a)$ .

We now show the remaining informations given in Tables 1 and 2. In Table 1, the fact that the given generators are a (pseudo-)orthogonal family of  $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$  is straightforward, e.g.  $\frac{1}{d} \operatorname{tr}(L^* L) = \frac{1}{d} \operatorname{tr}(-L^2) = \frac{1}{d} \operatorname{tr}(-\operatorname{Id}) = -1$  or, in case  $(\mathbf{2}^{\mathbb{C}})$ ,  $\frac{1}{d} \operatorname{tr}(L^* J) = \frac{1}{d} \operatorname{tr}(-J) = 0$  as  $J$  is a complex structure, or as  $\operatorname{tr}(L^* J) = \operatorname{tr}(JL^*) = -\operatorname{tr}(L^* J)$ . The different signatures of  $A^+ \subset A$  in column 2 follow.

Then, standard calculations show that each algebraic possibility of Table 1 corresponds to exactly one conjugation class of action of  $\mathfrak{s}$  on  $E$ , which is given by the matricial forms of  $L, J$  etc. in Table 2. This also shows the conditions implied on the signature of  $g$  in each case. To establish the three last columns of Table 2, we must check that the different (para)complex structures  $U$  announced are indeed the only ones. Take  $U \in \mathfrak{s}^-$ . Then  $U^2 = \pm \operatorname{Id} \Leftrightarrow U^* U = \mp \operatorname{Id} \Rightarrow \langle U, U \rangle = \mp 1$ . So, in cases  $(\mathbf{3})$ ,  $(\mathbf{3}')$ , and  $(\mathbf{3}^{\mathbb{C}})$ , the set of (para)kähler structures is included in the set announced in Table 2. The converse inclusion is an immediate calculation. All the other cases are left to the reader.

In cases  $(\mathbf{3})$ ,  $(\mathbf{3}')$ , and  $(\mathbf{3}^{\mathbb{C}})$ , after Proposition 1.23, the (pseudo-)Kähler manifold  $(\mathcal{M}, g, J)$  admits a non zero complex volume form so is Ricci flat. See also another brief proof in Theorem 5.1.

Finally, in §1.4 are built the (non empty) sets of germs of metrics inducing each case; in this paragraph, Proposition 1.27 and Remark 1.34 show Remark 1.15 above and hence the last assertion of the theorem.  $\square$

**1.21 Remark** In Claim 3 above, the use of the Bianchi identity, through Proposition 1.8, is necessary. Consider the case  $E = \mathbb{R}^{8p} \simeq \mathbb{H}^p \oplus \mathbb{H}^p$  and  $H' = \{u \in \operatorname{GL}_p(\mathbb{H})^2 : u = (u_1, {}^t \overline{u_1})\} \subset \operatorname{GL}_{8p}(\mathbb{R})$ . Then  $H'$  preserves the non degenerate real quadratic form  $(x_1, x_2) \mapsto {}^t \overline{x_1} \cdot x_2$  on  $E$ , and its action is orthogonally indecomposable (though  $H'$  cannot be a holonomy group). Now  $\mathfrak{gl}(E)^{b'} = (\operatorname{Id}_{\mathbb{H}^p} \cdot \mathbb{H})^2 \subset \operatorname{GL}_p(\mathbb{H})^2 \subset \operatorname{GL}_{8p}(\mathbb{R})$  and thus  $\mathfrak{gl}(E)^{b'} \simeq \mathbb{H} \oplus \mathbb{H}$ .

The following corollary of Theorem 1.11 may be noticed.

**1.22 Corollary** *A metric  $g$  admits parallel self adjoint complex structures exactly in cases  $(\mathbf{1}^{\mathbb{C}})$ ,  $(\mathbf{2}^{\mathbb{C}})$  and  $(\mathbf{3}^{\mathbb{C}})$ , and they are:  $\{\pm \underline{J} + N; N \in \mathfrak{n}_0 \text{ and } N\underline{J} = -\underline{J}N\}$ .*

**Proof.** Suppose that some  $\underline{J}_0 \in \mathfrak{e}^+$  satisfies  $\underline{J}_0^2 = -\operatorname{Id}$ . Take the decomposition  $\underline{J}_0 = S + N$  with  $S \in \mathfrak{s}^+$  and  $N \in \mathfrak{n}^+$ . By Lemma 1.18, and as the minimal polynomial of  $\underline{J}_0$  is  $X^2 + 1$ , irreducible,  $S^2 = -\operatorname{Id}$  so we are in case  $(\mathbf{1}^{\mathbb{C}})$ ,  $(\mathbf{2}^{\mathbb{C}})$  or  $(\mathbf{3}^{\mathbb{C}})$  and  $S = \pm \underline{J}$ . Now  $-\operatorname{Id} = \underline{J}_0^2 = (\underline{J} + N)^2 = -\operatorname{Id} + JN + NJ + N^2$ . By Proposition 1.8,  $JN - NJ \in \mathfrak{n}_0$ , so  $N(2J + N) = JN + NJ + N^2 - (JN - NJ) = -(JN - NJ) \in \mathfrak{n}_0$ . By Lemma 1.18,  $((2J + N)^2 + 4\operatorname{Id})^k = 0$  for some  $k$ , so  $2J + N$  is invertible, so  $N \in \mathfrak{n}_0$ , and as then  $N^2 = 0$ ,  $N \in \{U \in \mathfrak{n}_0; JU = -UJ\}$ .  $\square$

Finally, it may be useful to sum up and write down the different possible parallel tensors.

**1.23 Proposition** *In each case of Theorem 1.11, the metric admits the nondegenerate parallel multilinear or sesquilinear forms given in Table 3.*

**Proof.** Some lines of Table 3 require a brief checking.

(1) We must prove that any  $U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$  is nondegenerate. We read in Table 1 that any  $U \in \mathfrak{s}^+ \setminus \{0\}$  is, so its minimal polynomial  $\mu$  is not divisible by  $X$ ; by Lemma 1.18, neither is the minimal polynomial of  $U + N$  for any  $N \in \mathfrak{n}$ , so the result.

parallel tensor	exists in cases	parametrised by	given as
Pseudo-Riemannian metric	all	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$	$g(\cdot, U\cdot)$
Symplectic form	all except (1) and (1 <sup>C</sup> )	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-$	$g(\cdot, U\cdot)$
“Complex Riemannian” metric	(1 <sup>C</sup> ), (2 <sup>C</sup> ), (3 <sup>C</sup> )	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$ such that $U\underline{J} = \underline{J}U$	$\underline{g}_U = g(\cdot, U\cdot) + ig(\cdot, \underline{J}U\cdot)$
Hermitian (pseudo-) Kähler metric w. r. to some $J \in \mathfrak{s}^-$	(2), (2 <sup>C</sup> ), (3), (3'), (3 <sup>C</sup> )	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$ such that $UJ = JU$	$h_U = g(\cdot, U\cdot) + ig(\cdot, JU\cdot)$
$\underline{J}$ -complex symplectic form	(2 <sup>C</sup> ), (3 <sup>C</sup> )	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-$ , such that $N\underline{J} = \underline{J}N$	$\underline{\omega}_U = g(\cdot, U\cdot) + ig(\cdot, \underline{J}U\cdot)$
$J$ -complex symplectic form	(3), (3'), (3 <sup>C</sup> )	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-$ , such that $UJ = -JU$	$\omega_U = g(\cdot, U\cdot) + ig(\cdot, JU\cdot)$
Non null $\underline{J}$ -complex volume form	(1 <sup>C</sup> ), (2 <sup>C</sup> ), (3 <sup>C</sup> )	associated with $\underline{g}_U$ above	
Non null $J$ -complex volume form	(3), (3'), (3 <sup>C</sup> )	equal to $\omega_U^{\wedge(d/4)}$ with $\omega_U$ as above	

Table 3: *The real and complex parallel tensors existing in the different cases.* In cases (3), (3') and (3<sup>C</sup>),  $(\mathfrak{s}^-)^*$  is the complement of the isotropic cone in  $\mathfrak{s}^-$ . The real part of  $h_U$  is a (pseudo-)Riemannian metric, its imaginary part is a 2-form of type (1,1).

(2) First, if some nondegenerate alternate form is parallel for a torsion-free connection, it is closed, thus a symplectic form. Next, to get  $\mathfrak{e}^{*-} = \mathfrak{s}^{*-} \times \mathfrak{n}^-$ , proceed as in (i) above.

(3) If  $J$  is a parallel complex structure (self- or skew-adjoint), then possible nondegenerate complex bilinear forms are in correspondence with the  $g(\cdot, U\cdot) - ig(\cdot, V\cdot)$  such that (check the computations)  $\ker U \cap \ker V = \{0\}$ ,  $V = UJ$ ,  $U^* = U$  and  $V^* = V$ . By Proposition 1.2, the first condition implies that  $U \notin \mathfrak{n}$  or  $V \notin \mathfrak{n}$ , so by Lemma 1.18 and the argument given in (1) here, that  $U$  or  $V$  is nondegenerate, hence both. Now if  $J^* = -J$ , the relations give that  $UJ = -JU$  and  $VJ = -JV$ . As  $U^* = U$ , by Proposition 1.8, for any  $m \in \mathcal{M}$  and  $x, y \in T_m \mathcal{M}$ ,  $R(x, y)(UJ - JU) = 0$ . But  $UJ - JU = 2UJ$  is nondegenerate, this would imply that  $\mathcal{M}$  is flat. So  $J$  may only be self adjoint, we denote it by  $\underline{J}$ , and this time  $U\underline{J} = \underline{J}U$ . After table 1 and Lemma 1.18, the existence of such a  $\underline{J}$  leads to the announced form of  $\mathfrak{s}^+$ . The rest follows from the same reasoning as in (1).

(4)-(6) The reasoning is entirely similar to (3).

(7) If some parallel  $\underline{J}$  exists, so some complex Riemannian metric  $\underline{g}$ , take  $(e_i)_{i=1}^{d/2}$  some field of  $\underline{g}$ -orthonormal complex basis, and  $\nu = e_1^* \wedge \dots \wedge e_{d/2}^*$ . As  $\underline{g}$  is parallel, so is  $\nu$ .  $\square$

## 1.4 The space of germs of metrics realising each form of $\mathfrak{s}$

**1.24 Reminder** As it is well known and easy to check, metrics such that  $\mathfrak{s}$  is of type (1<sup>C</sup>) are the real parts of complex Riemannian metrics *i.e.* of holomorphic, non degenerate  $\mathbb{C}$ -bilinear forms on complex manifolds  $(\mathcal{M}, \underline{J})$ .

As it is also well known, germs of (pseudo-)Kähler metrics (type (2)) are parametrised by a Kähler potential  $u$ , which is a real function:

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}. \quad (\mathbf{a})$$

Similarly, germs of para Kähler metrics (type  $(\mathbf{2}')$ ) are parametrised by a para Kähler potential (see *e.g.* §2 of [2]). The supplementary distributions  $\ker(L - \text{Id})$  and  $\ker(L + \text{Id})$  are integrable and generate  $g$ -totally isotropic supplementary foliations. Take  $((x_i)_{i=1}^{d/2}, (y_i)_{i=1}^{d/2})$  coordinates adapted to this local decomposition. Then the metrics of type  $(\mathbf{2}')$  depend on a real potential  $u$  through:

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = \frac{\partial^2 u}{\partial x_i \partial y_j}. \quad (\mathbf{b})$$

A metric of type  $(\mathbf{2}^{\mathbb{C}})$  is given by the complexification of  $(\mathbf{a})$  or  $(\mathbf{b})$ , indifferently, *i.e.* with a complex potential  $u$  and, in  $(\mathbf{a})$ , with complex variables replacing the real and imaginary parts of the  $z_i$  or, in  $(\mathbf{b})$ , with complex variables  $(x_i)_i$  and  $(y_i)_i$ .

**1.25 Remark** Be careful however that a manifold  $(\mathcal{M}, g)$  of type  $(\mathbf{2})$  or  $(\mathbf{2}^{\mathbb{C}})$  has to be complex, hence in particular real analytic, whereas one of type  $(\mathbf{2}')$  may be only smooth.

**1.26 Remark** We recall also that the “complex Riemannian” metrics defined in Table 3 in cases  $(\mathbf{1}^{\mathbb{C}})$ ,  $(\mathbf{2}^{\mathbb{C}})$  and  $(\mathbf{3}^{\mathbb{C}})$  are holomorphic with respect to the self adjoint complex structure  $\underline{J}$ . Check that, if  $z_j = x_j + iy_j$  are complex coordinates,  $\frac{\partial}{\partial y_j} g_{k,l} = i \frac{\partial}{\partial x_j} g_{k,l}$  for all  $k, l$ .

**1.27 Proposition** *A generic metric of type  $(\mathbf{1})$ ,  $(\mathbf{2})$ ,  $(\mathbf{2}')$ ,  $(\mathbf{1}^{\mathbb{C}})$  or  $(\mathbf{2}^{\mathbb{C}})$  has the holonomy algebra given in Remark 1.15. More precisely, if the 2-jet at the origin of some metric of the wished type satisfies some dense open condition among such 2-jets, then its holonomy is algebra is as in Remark 1.15. In particular, those holonomy groups are obtained on a dense open subset, for the  $C^2$  topology, of the corresponding metrics.*

**Proof.** It is standard, but as we will generalise it in Theorem 3.2, we recall it here.

At the origin, we take normal coordinate vectors  $(X_i)_{i=1}^d$ ; moreover such that  $X_{i+1} = JX_i$  or  $X_{i+1} = LX_i$  for  $i$  odd, in case  $(\mathbf{2})$  or  $(\mathbf{2}')$ . So for any coordinate vectors  $U, V$ ,  $D_U V = 0$  at 0. By a straightforward calculation, for any coordinate vectors  $A, B, U, V$  at the origin:

$$g(R(A, B)U, V) = \frac{1}{2}(A.U.(g(B, V)) - B.U.(g(A, V)) - A.V.(g(B, U)) + B.V.(g(A, U))).$$

So in case  $(\mathbf{1})$ ,  $g(R(X_i, X_j)|_0 \cdot, \cdot)$  is the alternate part of the bilinear form:

$$\beta_{i,j} : (U, V) \mapsto X_i.U.g(X_j, V) - X_j.U.g(X_i, V).$$

The  $\beta_{i,j}$  depend on the second derivatives of the coefficients of  $g$  at 0, which are free in normal coordinates. So, on a dense open subset of the 2-jets of metrics, their alternate parts are linearly independent and span a  $\frac{d(d-1)}{2}$ -dimensional space. Now  $\dim \mathfrak{o}_d(\mathbb{R}) = \frac{d(d-1)}{2}$ , this algebra is hence generically spanned.

In case  $(\mathbf{2})$  we set, for  $j$  odd,  $Z_{\frac{j+1}{2}} = X_j - iX_{j+1}$  and  $\bar{Z}_{\frac{j+1}{2}} = X_j + iX_{j+1}$  in  $T^{\mathbb{C}}\mathcal{M}$ . The  $R(Z_i, Z_j)$  and  $R(\bar{Z}_i, \bar{Z}_j)$  vanish, and the  $R(Z_i, \bar{Z}_j)$  vanish when evaluated on  $\Lambda^2 T^{1,0}\mathcal{M}$  or  $\Lambda^2 T^{0,1}\mathcal{M}$ . So  $R$  is determined at 0 by the following  $\beta_{i,j}$ :

$$\beta_{i,j} : (Z_k, \bar{Z}_l) \mapsto g(R(Z_i, \bar{Z}_j), Z_k, \bar{Z}_l) = \frac{1}{2}(-\bar{Z}_j.Z_k.(g(Z_i, \bar{Z}_l)) - Z_i.\bar{Z}_l.(g(\bar{Z}_j, Z_k))),$$

thus by the fourth derivatives of the Kähler potential  $u$ . Those are free in normal coordinates, so on a dense open subset of the 2-jets of metrics, the  $(\beta_{i,j})_{i,j=1}^{d/2}$  are linearly independent hence span a  $(\frac{d}{2})^2$ -dimensional space. As  $\dim \mathfrak{u}_{d/2} = \frac{d^2}{4}$ , this algebra is generically spanned.

For **(2')**, replace the  $(Z_i, \overline{Z}_i)_{i=1}^{d/2}$  by the  $(X_{2i-1}, X_{2i})_{i=1}^{d/2}$ , and  $\mathfrak{u}_{d/2}$  by  $\mathfrak{gl}_{d/2}(\mathbb{R})$ .

For types **(1<sup>C</sup>)** and **(2<sup>C</sup>)**,  $R$  is  $\underline{J}$ -complex; repeat the proofs in complex coordinates.  $\square$

Now we describe the space of germs of metrics of type **(3)**, **(3')** and **(3<sup>C</sup>)**. It is classical for type **(3)** (hyperkähler), the other cases are only an adaptation of the argument.

**1.28 Notation** Take  $\varepsilon \in \{-1, 1\}$  and  $\delta \in \mathbb{N}^*$ . We denote by  $\mathcal{G}_\varepsilon$  the set of germs at 0 of triples  $(g, J, U)$  with  $g$  a (pseudo-)Riemannian metric on  $\mathbb{R}^d = \mathbb{R}^{4\delta}$  and  $J$  and  $U$  two  $g$ -skew adjoint parallel endomorphisms fields such that  $\varepsilon U^2 = -J^2 = \text{Id}$ , and  $JU = -UJ$ . We denote by  $\mathcal{G}_\mathbb{C}$  the set of germs at 0 of triples  $(g, J, U)$  with  $g$  a complex Riemannian metric on  $\mathbb{C}^{4\delta}$  and similar  $J$  and  $U$  (with *e.g.*  $\varepsilon = -1$ , but this makes no difference on  $\mathbb{C}$ ).

Using Cartan-Kähler theory, we parametrise  $\mathcal{G}_\varepsilon$  and  $\mathcal{G}_\mathbb{C}$  in the real analytic category. We proceed as R. Bryant did in [11] §2.5 pp. 122–126 for hyperkähler metrics *i.e.* for  $\varepsilon = -1$ , detailing slightly the calculations to show that the case  $\varepsilon = 1$  works alike, and to allow a generalisation in Theorem 3.2. The complex case follows.

**1.29 Remark/Notation** Let  $\omega_0$  be some complex symplectic form on some open set  $\mathcal{O}$  of  $\mathbb{C}^{2\delta}$ . Then any 2-form  $\omega$  of type (1,1), real, may be written as  $\omega = \Im(\omega_0(\cdot, U_\omega \cdot))$ , with  $U_\omega$  an  $\omega_0$ -self adjoint complex antimorphism field. The correspondence is bijective between such forms  $\omega$  and such  $U_\omega$ , so we use this notation  $U_\omega$  in the following.

**1.30 Remark** The set  $\mathcal{G}_\varepsilon$  is in bijection with the set  $\mathcal{G}'_\varepsilon$  of germs of couples  $(\omega_0, \omega)$ , with  $\omega_0$  a complex symplectic form on  $\mathbb{C}^{2\delta}$  and  $\omega$  a closed 2-form of type (1,1), real, such that  $U_\omega^2 = \varepsilon \text{Id}$ , through the following.

– Let  $(g, J, U)$  be given. Then set  $\mathbb{C}^{2\delta} := (\mathbb{R}^{4\delta}, J)$ ,  $\omega_0 := g(\cdot, U \cdot) + ig(\cdot, JU \cdot)$  and  $\omega := \varepsilon g(\cdot, J \cdot) = \Im(\omega_0(\cdot, U \cdot))$ . As  $DJ = DU = 0$ , immediately  $d\omega_0 = d\omega = 0$ .

– Let  $(\omega_0, \omega)$  be given. Then set  $(\mathbb{R}^{4\delta}, J) := (\mathbb{C}^{2\delta}, i)$ ,  $g := -\varepsilon \omega(\cdot, i \cdot)$  and  $U := U_\omega$ . As  $d\omega_0 = d\omega = 0$ , a standard calculation gives  $DJ = DU = 0$  (see *e.g.* [20] §11.2).

Here is the interest of this new point of view. After the Darboux theorem, up to a biholomorphism of  $\mathbb{C}^{2\delta}$ ,  $\omega_0$  may be considered as the canonical symplectic form  $\sum_{j=1}^\delta dz_i \wedge dz_{\delta+i} = \frac{1}{2} {}^t dz \wedge \Omega_0 \wedge dz$ , with  $dz$  the column  $(dz_i)_{i=1}^{2\delta}$  and  $\Omega_0$  the matrix  $\begin{pmatrix} 0 & I_\delta \\ -I_\delta & 0 \end{pmatrix}$ . So *from now on, we consider that  $\omega_0$  is this canonical form*. Then the elements of  $\mathcal{G}_\varepsilon$ , considered up to diffeomorphism of  $\mathbb{R}^{4\delta}$ , are in bijection with those of  $\mathcal{G}'_\varepsilon$ , considered up to symplectomorphism of  $(\mathbb{C}^{2\delta}, \omega_0)$ . Now we may use Cartan-Kähler theory to describe  $\mathcal{G}'_\varepsilon$ .

**1.31 Notation** Set  $V := \text{Mat}(U)$ ,  $U$  is an antimorphism so  $U(z) = V \cdot \overline{z}$ . As  $\omega_0(U \cdot, \cdot) = -\omega_0(\cdot, U \cdot)$ , we get  $\Omega_0 V = -{}^t \overline{V} \Omega_0$ . A 2-form  $\omega$  is in  $\mathcal{G}'_\varepsilon$  if and only if it is closed and:

$$\omega = \Im(\omega_0(\cdot, U \cdot)) = \frac{1}{2i} {}^t dz \wedge \Omega_0 V \wedge d\overline{z} \text{ with } V \overline{V} = \varepsilon \text{Id}$$

*i.e.*, setting  $H := -\Omega_0 V$ , if and only if:

$$\omega = \frac{i}{2} {}^t dz \wedge H \wedge d\overline{z} \text{ with } {}^t H = \overline{H} \text{ and } \overline{H} \Omega_0 H = -\varepsilon \Omega_0.$$

Let  $\mathcal{H}_\varepsilon \subset M_{2\delta}(\mathbb{C})$  be the space of such matrices  $H$ . The  $(1,1)$ -forms  $\omega$  such that  $U_\omega^2 = \varepsilon \text{Id}$  are exactly given by the functions  $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$ , through:  $\omega_H := \frac{i}{2} dz \wedge H(z) \wedge d\bar{z}$ . Denoting by  $(z, H)$  the points in  $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$ , such an  $\omega_H$  is *closed* if and only if the 3-form  $\lambda := {}^t dz \wedge dH \wedge d\bar{z}$  vanishes along the graph  $\mathcal{S}$  of  $H$ . So we are looking for the integral manifolds  $\mathcal{S}$  of the exterior differential system  $\mathbf{I} = \langle \lambda \rangle$  on  $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$ , with the independence condition that  $dz_1 \wedge \dots \wedge dz_{2\delta}$  never vanishes (*i.e.*  $\mathcal{S}$  is the graph of some  $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$ ). Then the Cartan-Kähler theorem parametrises  $\mathcal{G}'_\varepsilon$ , hence  $\mathcal{G}_\varepsilon$ , providing:

**1.32 Proposition** *The elements of  $\mathcal{G}_\varepsilon$ , considered up to diffeomorphism, are parametrised by  $\frac{d}{2} = 2\delta$  real analytic functions of  $2\delta + 1$  real variables. Those of  $\mathcal{G}_\mathbb{C}$ , up to biholomorphism, are parametrised by  $\frac{d}{4} = 2\delta$  holomorphic functions of  $2\delta + 1$  complex variables.*

**1.33 Remark** The generality of the elements of  $\mathcal{G}_\varepsilon$  and  $\mathcal{G}_\mathbb{C}$  ensures that their corresponding algebra  $\mathfrak{s}$  is indeed, generically, in cases **(3)**, **(3')** or **(3<sup>C</sup>)** (and *e.g.* not the full  $\text{End}(T\mathcal{M})$ ). In fact, their holonomy group itself is generically that of Remark 1.15, see Remark 1.34.

**Proof.** The writing of  $\mathbf{I}$  in  $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$  does not depend on  $z$ , so we have only to perform Cartan's test on some arbitrary fibre  $\{z_0\} \times H$ , say with  $z_0 = 0$ . Moreover, over that point  $z_0$ , the symplectic group  $\text{Sp}(2\delta, \mathbb{C})$  acts transitively on  $\{\frac{i}{2} dz \wedge H \wedge d\bar{z}; H \in \mathcal{H}_\varepsilon\}$ , preserving  $\mathbf{I}$ , so we have only to perform Cartan's test at some specific element  $H_0 \in \mathcal{H}_\varepsilon$ , say:

- if  $\varepsilon = -1$ ,  $H_0 = I_{p,q,p,q} = \text{diag}(I_p, -I_q, I_p, -I_q)$  with  $p + q = n$ ,
- if  $\varepsilon = 1$ ,  $H_0 = iI_{n,n}$ .

*Remark.* As it appears in [11], the connected component  $\mathcal{H}_{-1}^{p,q}$  of  $I_{p,q,p,q}$  in  $\mathcal{H}_{-1} = \sqcup_{p+q=n} \mathcal{H}_{-1}^{p,q}$  is canonically isomorphic to  $\text{Sp}(n, \mathbb{C}) / \text{Sp}(p, q)$ . So choosing some function  $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_{-1}^{p,q}$  amounts to choosing a reduction to  $\text{Sp}(p, q)$ , which is a real form of  $\text{Sp}(n, \mathbb{C})$ , of the principal bundle  $\text{Sp}(n, \mathbb{C}) \times \mathbb{C}^{2\delta}$ . Similarly here,  $\mathcal{H}_1 \simeq \text{Sp}(n, \mathbb{C}) / \text{Sp}(n, \mathbb{R})$  so choosing some  $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_1$  is choosing a reduction of it to  $\text{Sp}(n, \mathbb{R})$ , which is another real form of  $\text{Sp}(n, \mathbb{C})$ .

Let us set  $\partial z_j = \partial x_j + i\partial y_j$ . If a subspace  $E$  of  $T_{m_0}\mathcal{M}$  is horizontal *i.e.* tangent to the factor  $\mathbb{C}^{2\delta}$ ,  $\lambda|_E = 0$  so  $E$  is an integral element of  $\mathbf{I}$ . Let us define  $(E_k)_{k=0}^{4\delta}$  by:

$$E_k = \text{span} \left( (e_j)_{j=1}^k \right) \quad \text{with, for } 1 \leq j \leq \delta: \quad e_j = (\partial x_j, 0)$$

$$\text{and } e_{\delta+j} = \left( \partial x_{\delta+j} + \frac{j-1}{\delta} \partial y_{\delta+j}, 0 \right), \quad \text{and for } 1 \leq j \leq 2\delta: \quad e_{2\delta+j} = (\partial y_j, 0).$$

Each  $E_k$  is horizontal so  $(E_k)_{k=0}^{4\delta}$  is an integral flag of  $\mathbf{I}$  at  $m_0$ . We classically set  $H(E_k) := \{v; \text{span}(v, E_k) \text{ is an integral element of } \mathbf{I}\}$ , and  $s_k := \text{codim}_{H(E_{k-1})} H(E_k)$  the  $k$ th. character of  $\mathbf{I}$  (indeed this flag is ordinary, as we will see). We will check:

(1) for all  $k$ ,  $s_k = k - 1$ , and  $s_k = 0$  for  $k > 2\delta + 1$ ,

(2)  $\dim V_{4\delta}(\mathbf{I}) \geq 2C_{2\delta+2}^3$ , with  $V_{4\delta}(\mathbf{I})$  the variety of integral elements of  $\mathbf{I}$  in the grassmannian  $G_{4\delta}(T(\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon))$ .

After Cartan's criterion,  $\dim V_{4\delta}(\mathbf{I}) \leq \sum_{k=1}^{4\delta} k s_k$ , and if equality holds then  $E_{4\delta}$  is ordinary. So here:

$$\dim V_{4\delta}(\mathbf{I}) \leq \sum_{k=1}^{4\delta} k s_k = \sum_{k=1}^{2\delta+1} k(k-1) = \frac{8}{3}\delta^3 + 4\delta^2 + \frac{4}{3}\delta = 2C_{2\delta+2}^3 \leq \dim V_{4\delta}(\mathbf{I}),$$

therefore  $E_{4\delta}$  is ordinary and after the Cartan-Kähler theorem,  $\mathbf{I}$  admits an integral manifold  $\mathcal{S}$  through  $(0, H_0)$  with  $T\mathcal{S} = E_{4\delta}$ , and the space of germs of integral manifolds passing by  $z_0$  depends on  $s_k$  functions of  $k$  variables, with  $s_k$  the last non vanishing character, so here  $2\delta$  functions of  $2\delta + 1$  variables. This parametrisation of the set  $\mathcal{G}_\varepsilon$  is done up to the choice of complex Darboux coordinates for  $\omega_0$ , and such coordinates depend on one symplectic generating function of  $2\delta$  variables. As  $2\delta < 2\delta + 1$ , this does not interfere and  $\mathcal{G}_\varepsilon$  itself is parametrised by  $2\delta$  functions of  $2\delta + 1$  variables, the proposition. We are left with showing **(1)** and **(2)**.

We introduce  $W_\varepsilon := T_{H_0}\mathcal{H}_\varepsilon$ , then:

$$W_1 = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}; a, b \in M_\delta(\mathbb{C}), {}^t a = \bar{a}, {}^t b = b \right\}$$

$$\text{and: } W_{-1} = \left\{ \begin{pmatrix} a & I_{p,q}b \\ \bar{b}I_{p,q} & -I_{p,q}\bar{a}I_{p,q} \end{pmatrix}; a, b \in M_n(\mathbb{C}), {}^t a = \bar{a}, {}^t b = b \right\}.$$

Then  $T_{m_0}\mathcal{M}_\varepsilon = T_0\mathbb{C}^{2\delta} \oplus W_\varepsilon \simeq \mathbb{C}^{2\delta} \oplus W_\varepsilon$  and the subset of the grassmannian  $G_{4\delta}(T_{m_0}\mathcal{M})$  on which the independence condition holds is canonically identified with  $(\mathbb{C}^{2\delta})^* \otimes W_\varepsilon$ .

**(1)** follows from the fact that for  $k > 2\delta$ ,  $H(E_k) = \mathbb{C}^{2\delta} \oplus \{0\}$ , and for  $1 \leq k \leq \delta$ :

–  $H(E_k) = \mathbb{C}^{2\delta} \oplus \{\Im a_{i,j} = 0 \text{ for } 1 \leq i < j \leq k\} \subset \mathbb{C}^{2\delta} \oplus W_\varepsilon$ , so  $\text{codim}_{H(E_{k-1})} H(E_k) = k - 1$ ,

–  $H(E_{n+k}) = \mathbb{C}^{2\delta} \oplus \{\Re a_{i,j} = \Re b_{i,j} = 0 \text{ for } 1 \leq i < j \leq k \text{ and } \Im b_{k,j} + \frac{k-1}{\delta} \Re b_{k,j} = 0 \text{ for } k \leq j \leq \delta\} \subset \mathbb{C}^{2\delta} \oplus W_\varepsilon$ , so  $\text{codim}_{H(E_{\delta+k-1})} H(E_{\delta+k}) = \delta + k - 1$ .

To check **(2)**, we introduce some notation. We denote the basis vectors  $(\partial x_i)_{i=1}^{2\delta}$  of  $\mathbb{C}^{2\delta}$  by  $((u_i)_{i=1}^k, (u'_i)_{i=1}^k)$  (the  $u_i$  and  $u'_i$  are  $\omega_0$ -dual), then  $(\partial y_i)_{i=1}^{2\delta} = ((Ju_i)_{i=1}^k, (Ju'_i)_{i=1}^k)$ . We denote by  $H^{(1)}$  a generic element of  $(\mathbb{C}^{2\delta})^* \otimes W_\varepsilon$ . If a function  $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$  with  $H(0) = H_0$  is such that  $dH|_0 = H^{(1)}$ , then, at 0,  $d\omega_H$  is determined by  $d\omega_H = \lambda|_{m_0}(H^{(1)} \cdot, H^{(1)} \cdot, H^{(1)} \cdot)$ , that we denote by  $\lambda_{H^{(1)}}$ . In concrete terms, for the calculations below,  $\lambda_{H^{(1)}}$  reads:

$$\lambda_{H^{(1)}}(u, v, w) = \omega_0(u, H^{(1)}(v) \cdot \bar{w}) + \omega_0(v, H^{(1)}(w) \cdot \bar{u}) + \omega_0(w, H^{(1)}(u) \cdot \bar{v}).$$

At  $(0, H_0)$ ,  $V_{4\delta}(\mathbf{I})$  is the set of the 1-jets of *closed* 2-forms  $\omega_H$  as wanted. An  $H^{(1)}$  is in  $V_{4\delta}(\mathbf{I})$  if and only if  $\lambda_{H^{(1)}} = 0$ , which may be written as the two following conditions:

- (a) for any three  $\{u, v, w\} \subset \{u_i, Ju_i\}_{i=1}^k$ ,  $\lambda_{H^{(1)}}(u^{(l)}, v^{(l)}, w^{(l)}) = 0$ ,
- (b) for any two  $\{u, v\} \subset \{u_i, Ju_i\}_{i=1}^k$ ,  $\lambda_{H^{(1)}}(u, u', v^{(l)}) = \lambda_{H^{(1)}}(v, v', u^{(l)}) = 0$ .

The parenthesised primes enable to denote several equations at once, so **(a)** consists of  $8C_{2\delta}^3$  equations and **(b)** of  $4C_{2\delta}^2$ . Now the equations of **(a)** are redundant. Indeed the reader may check the following. Take any  $H^{(1)}$  and any  $\{i, j, k\} \subset \llbracket 1, \delta \rrbracket$  and  $\{\alpha, \beta, \gamma\} \subset \{0, 1\}$  such that  $\sharp\{J^\alpha u_i, J^\beta u_j, J^\gamma u_k\} = 3$  (so,  $C_{2\delta}^3$  possibilities). Set  $(u, v, w) := (J^\alpha u_i, J^\beta u_j, J^\gamma u_k)$  and, in case  $\varepsilon = 1$ ,  $\eta_1 := (-1)^{\gamma-\beta}$ ,  $\eta_2 := (-1)^{\alpha-\gamma}$  and  $\eta_3 := (-1)^{\beta-\alpha}$ , and in case  $\varepsilon = -1$ ,  $\eta_1 := (-1)^{\gamma-\beta}(-1)^{\chi_{\{k \leq p\}} + \chi_{\{j \leq p\}}}$ ,  $\eta_2 := (-1)^{\alpha-\gamma}(-1)^{\chi_{\{i \leq p\}} + \chi_{\{k \leq p\}}}$ ,  $\eta_3 := (-1)^{\beta-\alpha}(-1)^{\chi_{\{j \leq p\}} + \chi_{\{i \leq p\}}}$ . We denote by  $\chi_P$  the characteristic function of the set  $P$ , equal to 1 on  $P$  and null elsewhere. Explicitly,  $\chi_{\{i \leq p\}} + \chi_{\{j \leq p\}}$  is even if and only if  $(i, j) \subset \llbracket 1, p \rrbracket^2 \cup \{p+1, \delta\}^2$ . Then the following sets of relations (say respectively **(i)**, **(ii)**, **(iii)** and **(iv)**) hold.

$$\begin{cases} \eta_1 \lambda_{H^{(1)}}(u', v, w) + \eta_2 \lambda_{H^{(1)}}(u, v', w) + \eta_3 \lambda_{H^{(1)}}(u, v, w') + \varepsilon \lambda_{H^{(1)}}(u', v', w') = 0 \\ \eta_1 \lambda_{H^{(1)}}(Ju', v, w) + \eta_2 \lambda_{H^{(1)}}(u, Jv', w) + \eta_3 \lambda_{H^{(1)}}(u, v, Jw') + \varepsilon \lambda_{H^{(1)}}(Ju', Jv', Jw') = 0 \\ \eta_1 \lambda_{H^{(1)}}(u, v', w') + \eta_2 \lambda_{H^{(1)}}(u', v, w') + \eta_3 \lambda_{H^{(1)}}(u', v', w) + \varepsilon \lambda_{H^{(1)}}(u, v, w) = 0 \\ \eta_1 \lambda_{H^{(1)}}(u, Jv', Jw') + \eta_2 \lambda_{H^{(1)}}(Ju', v, Jw') + \eta_3 \lambda_{H^{(1)}}(Ju', Jv', w) + \varepsilon \lambda_{H^{(1)}}(u, v, w) = 0. \end{cases}$$



So the  $8C_{2\delta}^3$  linear forms of the type  $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u^{(i)}, (J)v^{(i)}, (J)w^{(i)})$  are linearly dependent, through the  $4C_{2\delta}^3$  equations above. In turn, those equations are linearly independent. Counting the number of primes appearing in them, one sees that equations of type (i) and (ii) on the one hand, and of type (iii) and (iv) on the other hand, span subspaces in direct sum. Now any dependence relation would involve some fixed triple  $(i, j, k)$ . For such a fixed triple, equations of type (i) may be seen as expressing the forms  $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u'_i, (J)u'_j, (J)u'_k)$  as combination of the other ones, and then equations of type (i)-(ii) as expressing the forms  $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u'_i, (J)u_j, (J)u_k)$  as combination of the other ones. Equations of type (iii) and (iv) are similar, so all the  $4C_{2\delta}^3$  equations are independent, and the  $8C_{2\delta}^3$  forms span a space of dimension  $\leq 8C_{2\delta}^3 - 4C_{2\delta}^3 = 4C_{2\delta}^3$ . So (a) and (b) consist of not more than  $4C_{2\delta}^3 + 4C_{2\delta}^2 = 4C_{2\delta+1}^3$  independent equations, so  $\dim V_{4\delta}(\mathbf{I}) \geq \dim[\mathbb{C}^{2\delta} \otimes W_\varepsilon] - 4C_{2\delta+1}^3 = (4\delta) \cdot (2\delta + \delta) - 4C_{2\delta+1}^3 = 2C_{2\delta+2}^3$ . This is (2).

Let us finally deal with  $\mathcal{G}_{\mathbb{C}}$ . All that precedes may be considered as involving only real variables: whenever complex variables  $x + iy$  appear, consider them as real matrices  $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ . With that point of view, complexifying everything *i.e.* replacing everywhere the real entries  $x, y$  by complex numbers amounts to parametrise  $\mathcal{G}_{\mathbb{C}}$ ; so reproducing the same reasoning gives the proposition for  $\mathcal{G}_{\mathbb{C}}$ .  $\square$

**1.34 Important Remark** Among real analytic germs of metrics with holonomy  $H$  included in  $H_0 = \mathrm{Sp}(p, q)$ ,  $H_0 = \mathrm{Sp}(2\delta, \mathbb{R})$  or  $H_0 = \mathrm{Sp}(2\delta, \mathbb{C})$ , corresponding to cases (3), (3') and (3''), a dense open subset for the  $C^2$  topology has its holonomy equal to  $H_0$ . Indeed, the first prolongation  $\mathbf{I}^{(1)}$  of the ideal  $\mathbf{I}$  satisfies also Cartan's criterion; this enables to show that any 2-jet of metric, integrable at the order 1 and such that  $\{R(X, Y); X, Y \in T_0\mathcal{M}\} \subset \mathfrak{h}_0$ , is the 2-jet of a metric with holonomy included in  $H_0$ . The reasoning is presented, in the case  $H = G_2$ , in Proposition 3 p. 556 of [10]. It may be adapted here, as indicated in [11] §2.5 p. 126. So as, among such 2-jets, those satisfying  $\{R(X, Y); X, Y \in T_0\mathcal{M}\} = \mathfrak{h}_0$  are generic, we get the result.

## 2 A way to understand n: parametrising metrics making parallel a nilpotent structure

### 2.1 Some notation and remarks about nilpotent endomorphisms

We found useful to sum up here standard facts, in a presentation of our own. This makes Part 2 more self-contained and allows to introduce, at its beginning, some coherent notation used all along afterwards. First, let  $N$  be in  $\mathrm{End}(\mathbb{R}^d)$ , nilpotent of index  $n$ .

We set  $F^{a,b} = \mathrm{Im} N^a \cap \ker N^b$ . Though we do not use the  $(F^{a,b})_{b=1}^{n-a}{}_{a=n-1}^0$  explicitly here, we had them throughout in mind. They are ordered by inclusion as shown in Table 4.

**2.1 Notation (i)** The invariant factors of  $N$  are:

$$\underbrace{(X, \dots, X)}_{d_1 \text{ times}}, \underbrace{(X^2, \dots, X^2)}_{d_2 \text{ times}}, \dots, \underbrace{(X^n, \dots, X^n)}_{d_n \text{ times}} = ((X^a)_{k=1}^{d_a})_{a=1}^n,$$

for some  $n$ -tuple  $(d_a)_{a=1}^n$ . We call here the  $(d_a)_a$  the *characteristic dimensions* of  $N$ , see (ii) for a justification. We set  $D_a = \sum_{k=1}^a d_k$ , this notation will also be useful;  $D_0 := 0$ .

$$\begin{array}{ccccccc}
F^{n,*} = \text{Im } N^n = \{0\} & & & & & & \\
\cap & & & & & & \\
F^{n-1,1} & = & \text{Im } N^{n-1} & & & & \\
\cap & & & & & & \\
F^{n-2,1} & \subset & F^{n-2,2} & = & \text{Im } N^{n-2} & & \\
\cap & & \cap & & & & \\
F^{n-3,1} & \subset & F^{n-3,2} & \subset & F^{n-3,3} & = & \text{Im } N^{n-3} \\
\cap & & \cap & & & & \\
\vdots & & \vdots & & & & \ddots \\
\cap & & \cap & & & & \\
F^{*,0} \subset F^{0,1} & \subset & F^{0,2} & \subset & \dots & \subset & F^{0,n} = \text{Im } N^0 \\
\parallel & \parallel & \parallel & & & & \parallel \\
\{0\} = \ker N^0 & \ker N & \ker N^2 & & & & \ker N^n = \mathbb{R}^d
\end{array}$$

Table 4: The  $F^{a,b} = \text{Im } N^a \cap \ker N^b$ , defined for  $(a,b) \in \mathbb{N}^2$ . If some  $F^{a,b}$  does not appear in the table, *i.e.* if  $a \notin \llbracket n-b, n \rrbracket$ , it is equal to one of the  $F^{a',b'}$  appearing in it.

(ii) We denote by  $\pi$  the projection  $\mathbb{R}^d \rightarrow \mathbb{R}^d / \text{Im } N$ . Then for each  $a \in \llbracket 1, n \rrbracket$ ,  $d_a = \dim(\pi(\ker N^a) / \pi(\ker N^{a-1}))$ . For any  $(a,b)$ ,  $F^{a,b} / (F^{a,b-1} + F^{a+1,b})$  is canonically isomorphic, through  $N^a$ , to  $\pi(\ker N^{b-a}) / \pi(\ker N^{b-a-1})$ .

**2.2 Remark/Definition** Let us denote by  $\mathbb{R}[\nu]$  the real algebra generated by  $\nu$  satisfying the unique relation  $\nu^n = 0$ . In other words,  $\mathbb{R}[\nu] = \mathbb{R}[X]/(X^n) \simeq \mathbb{R}[N]$ . Setting  $\nu V := N(V)$  for  $V \in \mathbb{R}^d$  turns  $\mathbb{R}^d$  into an  $\mathbb{R}[\nu]$ -module. As such, it is isomorphic to  $\prod_{a=1}^n (\nu^{n-a} \mathbb{R}[\nu])^{d_a}$  *i.e.*  $d_1$  factors on which  $\nu$  acts trivially,  $d_2$  factors on which  $\nu$  is 2-step nilpotent *etc.* Notice that this isomorphism is not canonical, even up to an automorphism of each of the factors. We set  $D := D_n = \sum_{a=1}^n d_a$ . We define a  $D$ -tuple of vectors  $\beta = (X_i)_{i=1}^D$  to be an *adapted spanning family* of  $\mathbb{R}^d$  as an  $\mathbb{R}[\nu]$ -module if each  $(X_i)_{i=1+D_a}^{D_{a+1}}$ , pushed on the quotient, is a basis of  $\pi(\ker N^{a+1}) / \pi(\ker N^a)$ , see Notation 2.1. In other terms,  $\beta$  spans  $\mathbb{R}^d$  as an  $\mathbb{R}[\nu]$ -module and the only relation the  $X_i$  satisfy is  $\nu^a X_i = 0$  for  $D_{a-1} < i \leq D_a$ ; or: each  $(X_i)_{i=1+D_{a-1}}^{D_a}$  spans the factor  $(\nu^{n-a} \mathbb{R}[\nu])^{d_a}$  as an  $\mathbb{R}[\nu]$ -module. It is a basis if and only if the  $\mathbb{R}[\nu]$ -module  $\mathbb{R}^d$  admits bases *i.e.* it is free *i.e.*  $d_a = 0$  for  $a < n$ .

We denote by  $n(i)$  the nilpotence index of  $N$  on each submodule  $\langle X_i \rangle$ ; so  $n(i) = a$  for  $D_{a-1} < i \leq D_a$ . We denote by  $(X_i, (Y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$  the basis  $(X_i, (N^a X_i)_{a=1}^{n(i)-1})_{i=1}^D$  of  $\mathbb{R}^d$  as an  $\mathbb{R}$ -vector space.

Now let  $g$  be a symmetric bilinear form on  $\mathbb{R}^d$  such that  $g(N \cdot, \cdot) = g(\cdot, N \cdot)$ .

**2.3 Remark/Definition** For each  $a \in \llbracket 1, n \rrbracket$ , the symmetric bilinear form  $g(\cdot, N^{a-1} \cdot)$  is well defined on the quotient  $\pi(\ker N^a) / \pi(\ker N^{a-1}) \simeq \mathbb{R}^{d_a}$ . Indeed, if  $X \in \ker N^a$  and  $Y \in \text{Im } N$ , so  $Y = NZ$ , then  $g(X, N^{a-1} Y) = g(X, N^a Z) = g(N^a X, Z) = 0$ . We denote by  $(r_a, s_a)$  its signature;  $r_a + s_a \leq d_a$ . It is standard that the couple  $(N, g)$  is characterised up to conjugation by this family of dimensions and signatures, called here its *characteristic signatures*. See *e.g.* the elementary exposition [17]. The form  $g$  is non degenerate if and only if each  $g(\cdot, N^{a-1} \cdot)$  is, and then  $\text{sign}(g) = (\sum_{a=1}^n \lfloor \frac{a}{2} \rfloor d_a + \sum_{a \text{ odd}} r_a, \sum_{a=1}^n \lceil \frac{a}{2} \rceil d_a + \sum_{a \text{ odd}} s_a)$  with  $\lfloor \cdot \rfloor$  the floor function.

Assuming, by 2.4, the “ $\mathbb{R}[\nu]$ -module” viewpoint, Propositions 2.5 and 2.6 follow.

## 2.4 Proposition/Definition Set:

$$E = (\mathbb{R}^d, N) \simeq \prod_{a=1}^n (\nu^{n-a} \mathbb{R}[\nu])^{d_a}.$$

On  $E$ , the  $\mathbb{R}[\nu]$ -bilinear forms  $h$  are in bijection with the real forms  $g$  satisfying  $g(N \cdot, \cdot) = g(\cdot, N \cdot)$ , through:  $h = \sum_{a=1}^n \nu^{a-1} g(\cdot, N^{n-a} \cdot)$ . We call such an  $h$  the  $\mathbb{R}[\nu]$ -bilinear form associated with  $g$ , and  $g$  the real form associated with  $h$ . Be careful that  $g$  is not the real part of  $h$ , but the coefficient of the highest power  $\nu^{n-1}$  of  $\nu$  in  $h$ .

**2.5 Proposition/Definition** Let  $h$  be an  $\mathbb{R}[\nu]$ -bilinear form on  $\mathbb{R}^d$  as in 2.4. If  $\beta = (X_i)_{i=1}^D$  is an adapted spanning family (see 2.2) of  $\mathbb{R}^d$ ,  $\text{Mat}_\beta(h) = \sum_{a=0}^{n-1} \nu^a H_a \in \text{M}_D(\mathbb{R}[\nu])$  where:

–  $H_a = \begin{pmatrix} 0 & 0 \\ 0 & \check{H}^a \end{pmatrix}$ , the upper left null square block, of size  $D_{n-1-a}$ , corresponding to  $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a} X_i = 0\}$ ,

– the upper left block  $\check{H}_0^a$  of  $\check{H}^a$  of size  $d_{n-a}$ , corresponding to  $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a} X_i \neq N^{n-a} X_i = 0\}$  is of signature  $(r_{n-a}, s_{n-a})$  introduced in 2.3, hence of rank  $r_{n-a} + s_{n-a}$ .

We call the  $(r_a, s_a)_{a=1}^D$  the signatures of  $h$ . So if  $S \oplus \text{Im } N = \mathbb{R}^d$ ,  $(r_a, s_a)$  is the signature of the (well defined) form  $h_{n-a}$  on the quotient  $(S \cap \ker N^a) / (S \cap \ker N^{a-1})$ .

**Proof.** For the first point:  $\nu^a h(X, \cdot) = 0$  as soon as  $X \in \ker N^a$ .  $\square$

**2.6 Proposition** In Proposition 2.5, choosing an adequate  $\beta$ , we may take  $\check{H}^a$  null except  $\check{H}_0^a = I_{d_a, r_a, s_a} := \text{diag}(I_{r_a}, -I_{s_a}, 0_{d_a - r_a - s_a}) \in \text{M}_{d_a}(\mathbb{R}[\nu])$  for all  $a$ . So with such a  $\beta$ :

$$\text{Mat}_\beta(h) = \text{diag}(\nu^{n-a} I_{d_a, r_a, s_a})_{a=1}^n = \begin{pmatrix} \nu^{n-1} I_{d_1, r_1, s_1} & & \\ & \ddots & \\ & & \nu^0 I_{d_n, r_n, s_n} \end{pmatrix} \in \text{M}_D(\mathbb{R}[\nu]).$$

Each block  $\nu^{n-a} I_{d_a, r_a, s_a}$  corresponds to the factor:

$$(\nu^{n-a} \mathbb{R}[\nu])^{d_a} = \text{span}_{\mathbb{R}[\nu]} \{(X_i)_{D_{a-1} < i \leq D_a}\} = \text{span}_{\mathbb{R}[\nu]} \{X_i; N^{a-1} X_i \neq N^a X_i = 0\}$$

of  $\mathbb{R}^d$ . The  $\mathbb{R}[\nu]$ -conjugation class of  $h$  is given by the signatures of  $h$ , and  $h$  is non degenerate if and only if  $r_a + s_a = d_a$  for each  $a$ .

Finally, take  $N$  a nilpotent structure on  $\mathcal{M}$ .

**2.7 Notation** The distributions  $\text{Im } N^a$  and  $\ker N^a$  are integrable, we denote their integral foliations by  $\mathcal{I}^a$  and  $\mathcal{K}^a$  respectively and set  $\mathcal{I} := \mathcal{I}^1$ . **From now on,  $\mathcal{U}$  is an open neighbourhood of  $m$  on which those foliations are trivial.** We still denote by  $\pi$  the projection  $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{I}$ .

Saying that  $N$  is integrable is saying that there are coordinates  $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$  of  $\mathcal{U}$  such that at each point, the basis

$$\left( X_i, (Y_{i,a})_{a=1}^{n(i)-1} \right)_{i=1}^D = \left( \frac{\partial}{\partial x_i}, \left( \frac{\partial}{\partial y_{i,a}} \right)_{a=1}^{n(i)-1} \right)_{i=1}^D$$

is of the type given in Remark 2.2. In the following,  $X_i$  and  $Y_{i,a}$  denote  $\frac{\partial}{\partial x_i}$  and  $\frac{\partial}{\partial y_{i,a}}$ .

## 2.2 Preliminary : introducing a special class of functions

We introduce here some material which is a bit more general than our strict subject. Just afterwards, back to our germs of pseudo-Riemannian metrics with a parallel field of nilpotent endomorphisms, this material will simplify a lot the statements and above all make them natural, as well as their proofs.

We still denote by  $\mathbb{R}[\nu]$  the real algebra generated by  $\nu$  satisfying the unique relation  $\nu^n = 0$ . We now mimic the definition of a holomorphic function  $f$  from a manifold  $\mathcal{M}$  with a complex structure  $J$ , to  $\mathbb{C} = \mathbb{R}[i]$ . The latter is a function such that  $df \circ J = i df$ . Here our manifold  $\mathcal{M}$  is endowed with a “nilpotent structure”: an integrable field of endomorphisms such that  $N^{n-1} \neq N^n = 0$ . This leads to the following definition.

**2.8 Definition** *Let  $f$  be some differentiable function  $(\mathcal{M}, N) \rightarrow \mathbb{R}[\nu]$ . We say here that  $f$  is nilomorphic (for the nilpotent structure  $N$ ) if  $df \circ N = \nu df$ .*

**2.9 Notation** If  $\eta$  is a function or more generally a tensor with values in  $\mathbb{R}[\nu]$ , we denote by  $\eta_a \in \mathbb{R}$  its coefficient of degree  $a$  in its expansion in powers of  $\nu$ , so that:  $\eta = \sum_{a=0}^{n-1} \eta_a \nu^a$ .

**2.10 Example** The simplest example of such functions are “nilomorphic coordinates”, built once again similarly as complex coordinates  $z_j := x_j + iy_j$  on a complex manifold. Take the  $N$ -integral coordinates  $(x_i, (y_{i,a})_a)_i$  introduced on  $\mathcal{U}$  in Notation 2.7 and set:

$$z_i := x_i + \nu y_{i,1} + \nu^2 y_{i,2} + \dots + \nu^{n(i)-1} y_{i,n(i)-1} \in \mathbb{R}[\nu].$$

Then each  $\nu^{n-n(i)} z_i$  is nilomorphic. Indeed, take  $X_i$  any coordinate vector transverse to  $\text{Im } N$  and  $a \in \mathbb{N}$ . Then  $(N^a X_j) \cdot (\nu^{n-n(i)} z_i) = 0$  if  $i \neq j$  and  $(N^a X_i) \cdot (\nu^{n-n(i)} z_i) = \nu^{n-n(i)} \nu^a$  (it is immediate if  $N^a X_i \neq 0$ ; besides  $N^a X_i = 0$  if and only if  $a \geq n(i)$ , which ensures that both sides of the equality vanish simultaneously). In particular  $(N^a X_i) \cdot z_i = \nu^a (X_i \cdot z_i)$ .

**2.11 Definition/Notation** We now call the  $z_i$  of Example 2.10 themselves “nilomorphic coordinates” even if only the  $\nu^{n-n(i)} z_i$  are nilomorphic functions. The reason appears in Remark 2.24. Moreover we introduce the notation:

$$(\nu y_i) := \sum_{a=1}^{n(i)-1} \nu^a y_{i,a}, \quad \text{so that } z_i \text{ reads: } z_i = x_i + (\nu y_i).$$

The use of nilomorphic coordinates will be much alleviated. We also set:  $(\nu y) := ((\nu y_i)_{i=1}^D)$ .

**2.12 Remark** Definition 2.8 may be stated for functions with value in any  $\mathbb{R}[\nu]$ -module.

**2.13 Remark** A system of holomorphic coordinates provides an isomorphism between a small neighbourhood of any point  $m$  of  $(\mathcal{M}, J)$  onto a small neighbourhood of the origin in  $\mathbb{C}^K$ . So does a system of nilomorphic  $(\nu^{n-n(i)} z_i)_i$ , from a small neighbourhood of any point  $m$  of  $(\mathcal{M}, N)$  onto a small neighbourhood of the origin in some  $\mathbb{R}[\nu]$ -module  $\mathbb{M}$ . Notice the small following difference:  $\mathbb{M}$  is not free *i.e.*  $\mathbb{M} \not\cong (\mathbb{R}[\nu])^K$  in general, but  $\mathbb{M} \simeq \prod_a (\nu^a \mathbb{R}[\nu])^{K(a)}$ . See the previous section. This is linked to the  $\nu^{n-n(i)}$  factoring the coordinates  $z_i$ .

**2.14 Reminder** A tensor  $\theta$  on a foliated manifold  $(\mathcal{M}, \mathcal{F})$  is said to be *basic* for  $\mathcal{F}$ , or  $\mathcal{F}$ -basic, if it is everywhere, locally, the pull back by  $p : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{F}$  of some tensor  $\bar{\theta}$  of  $\mathcal{M}/\mathcal{F}$ .

We will need also the following auxiliary definition.

**2.15 Definition** Let  $\mathbb{M}$  be an  $\mathbb{R}[\nu]$ -module. A function  $\check{f} : (\mathcal{U}/\mathcal{I}) \rightarrow \mathbb{M}$  is said to be adapted (to  $N$ ) if for each  $a \in \llbracket 0, n-1 \rrbracket$ ,  $\nu^a \check{f}$  is  $\pi(\mathcal{K}^a)$ -basic i.e. constant along the leaves of  $\pi(\mathcal{K}^a)$ . If  $\mathbb{M} = \mathbb{R}[\nu]$ , this means that each coefficient  $\check{f}_a$  is  $\pi(\mathcal{K}^{n-1-a})$ -basic.

Similarly, a (multi)linear form  $\check{\eta}$  defined on  $\mathcal{U}/\mathcal{I}$  with values in  $\mathbb{M}$  is called adapted if each  $\nu^a \check{\eta}$  is  $\check{\mathcal{K}}^a$ -basic. If  $\mathbb{M} = \mathbb{R}[\nu]$ , this means that each coefficient  $\check{\eta}_a$  is  $\check{\mathcal{K}}^{n-1-a}$ -basic.

The main property of nilomorphic functions we will use is the following.

**2.16 Proposition** Let  $\mathbb{M}$  be an  $\mathbb{R}[\nu]$ -module and  $f \in C^{n-1}(\mathcal{U}, \mathbb{M})$ . Then  $f$  is nilomorphic for  $N$  if and only if, in any nilomorphic coordinates system  $(z_i)_{i=1}^D = (x_i + (\nu y_i))_{i=1}^D$ , it reads:

$$f = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^{\alpha}} (\nu y)^{\alpha},$$

where  $\check{f}$  is some adapted function of the coordinates  $(x_i)_{i=1}^D$  and where, classically:

$$\begin{aligned} \alpha \text{ is a multi-index } (\alpha_i)_{i=1}^D, \quad |\alpha| &:= \sum_{i=1}^D \alpha_i, \quad \alpha! := \prod_{i=1}^D \alpha_i! , \\ \frac{\partial^{|\alpha|} \check{f}}{\partial x^{\alpha}} &:= \left( \frac{\partial^{|\alpha_1|}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_D}}{\partial x_D^{\alpha_D}} \right) \check{f}, \text{ and } (\nu y)^{\alpha} := \prod_{i=1}^D (\nu y_i)^{\alpha_i}. \end{aligned}$$

**2.17 Remark** Proposition 2.16 is very similar to the fact that a function  $(\mathcal{M}, J) \rightarrow \mathbb{C}$  is holomorphic if and only if it is equal to a power series in the neighbourhood  $\mathcal{U}$  of any point. In the complex case, we may consider that the coordinates  $x_i$  and  $y_i$  parametrise the integral leaves of  $\text{Im } J$  (complicated manner to mean the whole  $\mathcal{U}$ ), and that a single point  $m$  is a manifold transverse to this leaf. Then  $f$  is holomorphic if and only if it reads, in any holomorphic coordinates system:

$$f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}} \right)_{|m} z^{\alpha}.$$

In the formula of Propostion 2.16 appear:

- instead of the  $z^{\alpha}$ , the  $(\nu y)^{\alpha}$ , which are the powers of the coordinates parametrising the integral leaves  $\mathcal{I}$  of  $\text{Im } N$  (this time, they are not the whole  $\mathcal{U}$ ),
- instead of the value and derivatives of  $f$  at the single point  $m$  (a “transversal” to  $\mathcal{U}$ ), the values and derivatives of  $\check{f}$ , which is  $f$  along the level  $\mathcal{T} := \{(\nu y) = 0\} = \{\forall i, (\nu y_i) = 0\}$ , a transversal to the leaves of  $\mathcal{I}$ .

So in the complex case, you choose the value and derivatives of a holomorphic function at some point (ensuring a convergence condition), the rest of the function is given by a power series. In the “nilomorphic” case, you choose the value of  $f$  along some transversal  $\mathcal{T}$  to  $\mathcal{I}$  (ensuring the “adaptation” condition 2.15), the rest of the function is given by a power series. Notice finally that, as  $\nu^n = 0$ , this series in powers of  $(\nu y)$  is not only an analytic function, but even a *polynomial* one, of degree  $n-1$ . So  $f$  is polynomial along the leaves of  $\mathcal{I}$ ; in particular, this means that this notion makes sense, in any  $N$ -nilomorphic coordinates system; see Example 2.26 for an explanatory point of view. Transversely to those leaves however,  $f$  may be only of class  $C^{n-1}$ .

**2.18 Remark** One of the interests of the development formula of Proposition 2.16 is that it holds even if the invariant factors of  $N$  have different degrees *i.e.* the  $\mathbb{R}[\nu]$ -module  $(T\mathcal{M}, N)$  is not free. This will be useful to build metrics making  $N$  parallel in such cases, *e.g.* as in Example 2.39; those cannot be deduced from the case where  $(T\mathcal{M}, N)$  is a free  $\mathbb{R}[\nu]$ -module.

**Proof of Proposition 2.16. (i) The “if” part.** Take  $f$  of the form given in the proposition,  $X_i$  any coordinate vector transverse to  $\text{Im } N$  and  $a \in \mathbb{N}^*$ . Let us check that  $(N^a X_i).f = \nu^a (X_i.f)$ . If  $\alpha = (\alpha_i)_{i=1}^D$ ,  $\alpha \pm 1_i$  stands for  $(\alpha_1, \dots, \alpha_{i-1}, \alpha_i \pm 1, \alpha_{i+1}, \dots, \alpha_D)$ .

$$\nu^a X_i. \left( \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) = \nu^a \frac{\partial}{\partial x_i} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha = \nu^a \frac{1}{\alpha!} \frac{\partial^{|\alpha|+1} \check{f}}{\partial x^{\alpha+1_i}} (\nu y)^\alpha.$$

As  $\check{f}$  is adapted (Def. 2.15),  $\nu^{n(i)} \check{f}$  is constant along the leaves of  $\mathcal{K}^{n(i)}$ . So, as  $X_i \in \ker N^{n(i)}$ ,

$$\begin{aligned} (N^a X_i). \left( \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) &= \chi_{\{a < n(i)\}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} \nu^a \alpha_i (\nu y)^{\alpha-1_i} \\ &= \chi_{\{a < n(i)\}} \nu^a \frac{1}{\alpha!} \frac{\partial^{|\alpha|+1} \check{f}}{\partial x^{\alpha+1_i}} (\nu y)^{\alpha'} \quad \text{with } \alpha' := \alpha - 1_i. \end{aligned}$$

As  $(N^a X_i). \left( \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) = 0$  if  $\alpha_i = 0$ , we get the following equality, which concludes:

$$(N^a X_i).f = \chi_{\{a < n(i)\}} \nu^a \sum_{\alpha'} \frac{1}{\alpha'!} \frac{\partial^{|\alpha'|+1} \check{f}}{\partial x^{\alpha'+1_i}} (\nu y)^{\alpha'} = \nu^a X_i.f.$$

**(ii) The “only if” part.** If  $f$  is nilomorphic, then its restriction  $\check{f}$  to the level  $\mathcal{T} = \{\forall i, (\nu y_i) = 0\}$ , as a function of the  $x_i$ , is adapted: if  $N^a X = 0$ ,  $X.(\nu^a \check{f}) = (N^a X).\check{f} = 0$ .

Therefore, denoting by  $\check{f}'$  the restriction of  $f$  to  $\mathcal{T}$ , viewed as a function of the  $x_i$ , the function  $f'$  defined as  $f' := \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}'}{\partial x^\alpha} (\nu y)^\alpha$  is nilomorphic by (i).

*Claim.* If  $g : \mathcal{U} \rightarrow \mathbb{M}$  is a nilomorphic function vanishing on  $\mathcal{T}$ , then  $g = 0$ .

Applying the claim to  $g = f - f'$  gives that  $f = f'$ , so that  $f$  is of the wanted form. We are left with proving the claim. For any  $i$  and  $a > 0$ , as  $g$  is nilomorphic,  $(N^a X_i).g = \nu^a (X_i.g)$ . In the quotient  $\mathbb{M}/\nu\mathbb{M}$ , this reads  $(N^a X_i).[g]_{\mathbb{M}/\nu\mathbb{M}} = 0$  so  $[g]_{\mathbb{M}/\nu\mathbb{M}} \equiv 0$ . This gives rise to an induction:  $X_i.g \in \nu\mathbb{M}$  so in  $\mathbb{M}/\nu^2\mathbb{M}$ ,  $(N^a X_i).[g]_{\mathbb{M}/\nu^2\mathbb{M}} = 0$ , hence  $[g]_{\mathbb{M}/\nu^2\mathbb{M}} \equiv 0$ . By induction we get  $[g]_{\mathbb{M}/\nu^b\mathbb{M}} \equiv 0$  for all  $b$  and finally  $g = 0$ .  $\square$

The tangent spaces  $(T_m \mathcal{M}, N)$  are  $\mathbb{R}[\nu]$ -modules, through:  $\nu.X := NX$ . Considering the action of  $N$  as that of a scalar leads naturally to introduce the “nilomorphic” version of the tensors: all definitions and proofs are natural,  $\mathbb{R}[\nu]$ -linearity replacing  $\mathbb{R}$ -linearity. Now,  $z_i = x_i + (\nu y_i)$  are local nilomorphic coordinates and we set, after Notation 2.7,  $X_i := \frac{\partial}{\partial x_i}$ ,  $Y_{i,a} := \frac{\partial}{\partial y_{i,a}}$ . Then  $(X_i)_{i=1}^D$  is an adapted spanning family of each  $T_m \mathcal{M}$ , see Definition 2.2.

**2.19 Remark** The adapted function  $\check{f}$  in Proposition 2.16 depends in general on the choice of the transversal  $\mathcal{T} = \{(\nu y) = 0\}$ . We do not study this dependence here. The interested reader may look at the link with the expansion of functions in jet bundles indicated in Example 2.26 to understand a meaning of this dependence. We let however notice here that  $\check{f}_0$  is canonical *i.e.* does not depend on the choice of  $\mathcal{T}$ , and more generally, the value of  $\check{f}_a$  along each leaf of  $\pi(\mathcal{K}^{n-a})$  does not depend on it either, up to an additive constant.

**2.20 Definition/Proposition** A vector field  $V$  on  $(\mathcal{M}, N)$  is called nilomorphic if  $\mathcal{L}_V N = 0$ . Equivalently: in nilomorphic coordinates  $z_i = (x_i + (\nu y_i))_i$ ,  $V = \sum_i v_i X_i$  with nilomorphic functions  $v_i$ .

**Proof.** Any vector field reads  $V := \sum_i v_i X_i$  with  $v_i : \mathcal{M} \rightarrow \mathbb{R}[\nu]$ . Now  $\mathcal{L}_V N = 0$  if and only if, for any  $a$  and  $j$ ,  $[V, N^{a+1} X_j] = N[V, N^a X_j]$ . The  $(N^a X_i)_{a,i}$  commute, so  $[V, N^{a+1} X_j] = -\sum_i (\mathcal{L}_{N^{a+1} X_j} v_i) X_i$  and  $N[V, N^a X_j] = -\sum_i (\mathcal{L}_{N^a X_j} v_i) N X_i = \sum_i (\mathcal{L}_{N^a X_j} v_i) \nu X_i$ . Hence  $\mathcal{L}_V N = 0 \Leftrightarrow \forall i, j, a, \mathcal{L}_{N^{a+1} X_j} v_i = \nu \mathcal{L}_{N^a X_j} v_i$ , the result.  $\square$

**2.21 Definition/Proposition** Let  $\eta$  be some (multi)linear form on  $(\mathcal{M}, N)$ , with values in  $\mathbb{R}[\nu]$ . We say here that  $\eta$  is nilomorphic if:

(i) at each point, it is  $\mathbb{R}[\nu]$ -(multi)linear i.e.:

$$\forall m \in \mathcal{M}, \forall (V_i)_{i=1}^k \in T_m \mathcal{M}, \forall (a_i)_{i=1}^k \in \mathbb{N}^k, \eta(N^{a_1} V_1, \dots, N^{a_k} V_k) = \nu^{\sum_i a_i} \eta(V_1, \dots, V_k),$$

(ii)  $\mathcal{L}_{NV} \eta = \nu \mathcal{L}_V \eta$  for all nilomorphic vector field  $V$ .

If (i) is verified, then (ii) means that in nilomorphic coordinates  $z_i = x_i + (\nu y_i)$ , the coefficients  $\eta(X_{i_1}, \dots, X_{i_k})$  of  $\eta$  are nilomorphic functions (left to the reader).

**2.22 Remark** Notice that point (i) above implies that  $\nu^a \eta(V_1, \dots, V_k) = 0$ , in other words that  $\nu^{n-a} |\eta(V_1, \dots, V_k)|$ , as soon as some  $V_i$  is in  $\ker N^a$ . This means in particular that, setting  $\eta = \sum_{a=0}^{n-1} \nu^a \eta_a$  with real  $\eta_a$ , at each point  $m$ , each  $\eta_a$  is the pull back of a (multi)linear application  $(T_m \mathcal{M} / \ker N^{n-1-a})^k \rightarrow \mathbb{R}[\nu]$ .

Applying proposition 2.16 to the coefficients of any nilomorphic (multi)linear for  $\eta$  gives:

**2.23 Proposition** Let  $\eta$  be an  $\mathbb{R}[\nu]$ -(multi)linear form on  $\mathcal{U}$ . Then  $\eta$  is nilomorphic for  $N$  if and only if, introducing  $(z_i)_{i=1}^D$  as in 2.11 it reads:

$$\eta = \sum_{(i_1, \dots, i_k)} \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \tilde{\eta}_{i_1, \dots, i_k}}{\partial x^\alpha} (\nu y)^\alpha dz_{i_1} \otimes \dots \otimes dz_{i_k}$$

where the  $\tilde{\eta}_{i_1, \dots, i_k}$  are adapted functions of  $(x_i)_{i=1}^D$  with value in  $\nu^{n-\min_{l=1}^k n(i_l)} \mathbb{R}[\nu]$ .

**2.24 Remark** So, in the coordinates, it has to be noticed that the “elementary” nilomorphic multilinear forms are the  $\nu^{n-\min(n(i_1), \dots, n(i_k))} dz_{i_1} \otimes \dots \otimes dz_{i_k}$ . If  $k > 1$ , they may be written only with the  $z_i$  and not the  $\nu^{n-n(i)} z_i$ ; that is why we chose in 2.11 to define the former as “nilomorphic coordinates”. An expression of  $\eta$  using those elementary forms is:

$$\eta = \sum_{(i_1, \dots, i_k)} \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \tilde{\eta}_{i_1, \dots, i_k}}{\partial x^\alpha} (\nu y)^\alpha (\nu^{n-\min(n(i_1), \dots, n(i_k))} dz_{i_1} \otimes \dots \otimes dz_{i_k})$$

but then each  $\tilde{\eta}_{i_1, \dots, i_k}$ , with value in  $\mathbb{R}[\nu]/(\nu^{\min_{l=1}^k n(i_l)})$ , is such that  $\nu^{n-\min_{l=1}^k n(i_l)} \tilde{\eta}_{i_1, \dots, i_k}$  is adapted. So we will prefer the expression of Prop. 2.23, which is simpler, in the following.

The following result, which is now immediate, characterises the nilomorphic forms in terms of real ones.

**2.25 Definition/Proposition** Let  $\theta \in \Gamma(\otimes^k T^* \mathcal{M})$  be a real  $k$ -linear form on  $(\mathcal{M}, N)$ . We call it pre-nilomorphic if:

- (i) for any  $(V_j)_{j=1}^k$ , the  $\theta((V_j)_{j=1}^{i-1}, NV_i, (V_j)_{j=i+1}^k)$  are equal to each other, for all  $i$ ,
- (ii)  $\mathcal{L}_{NV}\theta = \mathcal{L}_V\theta(N \cdot, \cdot, \dots, \cdot)$  for all nilomorphic vector field  $V$ .

Then the  $\mathbb{R}[\nu]$ -valued  $k$ -linear form

$$\Theta := \sum_{a=0}^{n-1} \nu^a \theta(N^{n-1-a} \cdot, \cdot, \dots, \cdot)$$

is nilomorphic, and called the nilomorphic form associated with  $\theta$ . Conversely, any nilomorphic  $k$ -linear form  $\Theta = \sum_{a=0}^{n-1} \Theta_a \nu^a$ , with real  $\Theta_a$ , is associated, in this sense, with its coefficient  $\Theta_{n-1}$  — which is, necessarily, pre-nilomorphic.

The few comments below are unnecessary to understand the next section's theorem and proofs. The example is given because it is natural, but moreover it gives another point of view on nilomorphic functions. In that line, the remark following it makes a link with another work.

**2.26 Example** Natural manifolds with a nilpotent structure are the jet bundles  $J^n \mathcal{W}$  over some differentiable manifold  $\mathcal{W}$ . The fibre at some point  $m \in \mathcal{W}$  is  $\{f : ]-\varepsilon, \varepsilon[ \rightarrow \mathcal{W} ; \varepsilon > 0 \text{ and } f(0) = m\} / \sim$ , where  $f \sim g$  if in some neighbourhood of 0, then in all of them,  $\|f(t) - g(t)\| = o(t^n)$  when  $t \rightarrow 0$ . So  $J^0 \mathcal{W} = \mathcal{W}$  and  $J^1 \mathcal{W} = T\mathcal{W}$ . With each local chart  $\varphi = (x_i)_{i=1}^d : \mathcal{O} \rightarrow \mathbb{R}^d$  on some open set  $\mathcal{O}$  of  $\mathcal{W}$  is functorially associated a natural chart  $\tilde{\varphi} = ((x_{i,a})_{a=0}^n)_{i=1}^d : J^n \mathcal{O} \rightarrow \mathbb{R}^d$ , defined by:

$$\tilde{\varphi}([f]) = ((x_{i,a})_{a=0}^n)_{i=1}^d =: (\bar{x}_a)_{a=0}^n \text{ if } f(t) = \bar{x}_0 + t\bar{x}_1 + \dots + t^n \bar{x}_n + o(t^n).$$

A change of chart  $\theta$  on  $\mathcal{W}$  induces a change of chart  $\tilde{\theta}$  given by the action of the successive differentials of  $\theta$  up to order  $n$ , on the  $\bar{x}_a$ . The projections

$$\mathcal{W} \leftarrow J^1 \mathcal{W} \leftarrow J^2 \mathcal{W} \leftarrow \dots \leftarrow J^n \mathcal{W} \leftarrow \dots$$

endow each  $J^n \mathcal{W}$  with a flag of foliations  $\mathcal{K}^1 \subset \dots \subset \mathcal{K}^n$  where, in any chart of the type  $\tilde{\varphi}$ ,  $\mathcal{K}^a$  is given by the levels of  $(\bar{x}_b)_{b=0}^{n-a}$ . Now  $\mathbb{R}[X]/(X^{n+1})$  acts naturally on  $TJ^n \mathcal{W}$ , through the endomorphism  $N$  defined as follows. If the path  $(f + sg)_{s \in \mathbb{R}}$  represents, in some chart and at  $s = 0$ , some tangent vector to  $J^n \mathcal{W}$  at the point  $[f]$ ,  $N$  sends it to  $(f + sN(g))_{s \in \mathbb{R}}$ , with  $(N(g))(t) = tg(t)$ . This definition is consistent, all this is classical. At  $v = (\bar{v}_a)_{a=0}^n \in T_{[f]} J^n \mathcal{W}$ , in some chart of the type  $\tilde{\varphi}$ ,  $N$  reads:

$$N((\bar{v}_a)_{a=0}^n) = (0, (\bar{v}_a)_{a=0}^{n-1}).$$

For any  $n$ , in the coordinates, the matrix of  $N$  is in constant block-Jordan form, thus is integrable; its invariant factors are the  $d$ -tuple  $(X^{n+1}, \dots, X^{n+1})$ . So  $(J^n \mathcal{W}, N)$  is a manifold with a nilpotent structure. The foliation  $\mathcal{K}^a$  is the integral foliation of  $\ker N^a$ , which coincides here with  $\text{Im } N^{n+1-a}$ . Moreover, the submanifold  $\mathcal{T}_0$  of the jets of constant functions is a privileged transversal to the integral foliation  $\mathcal{I}$  of  $\text{Im } N$ . Conversely, any manifold  $\mathcal{M}$  with such a nilpotent structure  $N$ , i.e. with all its invariant factors of the same degree  $n+1$ , moreover endowed with some fixed transversal  $\mathcal{T}$  to  $\mathcal{I}$ , is locally modelled on the  $n^{\text{th}}$  jet bundle of the (local) quotient  $\mathcal{M}/\mathcal{I}$ . More explicitly, any nilomorphic coordinates



system of  $\mathcal{M}$  is exactly given by the choice of some transversal  $\mathcal{T}$  to  $\mathcal{I}$  and some chart  $\varphi$  of  $\mathcal{M}/\mathcal{I}$ . Now if  $\mathcal{T}$  is fixed on  $\mathcal{M}$ , the correspondence, given on the left in nilomorphic coordinates induced by some chart  $\varphi$  of  $\mathcal{M}/\mathcal{I}$ , and on the right in the chart  $\tilde{\varphi}$ :

$$\begin{array}{ccc} (\mathcal{M}, N) & \rightarrow & J^n(\mathcal{M}/\mathcal{I}) \\ \Psi : (x_i + (\nu y_i))_{i=1}^D = (x_i + \sum_{a=1}^n \nu^a y_{i,a})_{i=1}^D & \mapsto & \left[ (x_i + \sum_{a=1}^n t^a y_{i,a})_{i=1}^D \right] \end{array}$$

is independent of the choice of  $\varphi$ . This follows from the form of nilomorphic functions given by Proposition 2.16. Through this correspondence,  $(\mathcal{M}, \text{Diff}(\mathcal{M}, N, \mathcal{T}))$  is identified with  $J^n(\mathcal{M}/\mathcal{I})$  with the standard action of  $\text{Diff}(\mathcal{M}/\mathcal{I})$  on it. So in this sense, a manifold with a nilpotent structure the invariant factors of which are all of the same degree  $n+1$  is, locally, a jet bundle of order  $n$  where you “forgot” what is the submanifold of constant jets.

Introducing functions  $\tilde{u} : J^n \mathcal{W} \rightarrow \mathbb{R}[\nu] = \mathbb{R}[X]/(X^{n+1}) \simeq J_0^n \mathbb{R}$  is therefore natural; they are  $(N, \nu)$ -nilomorphic if and only if they represent, modulo some constant, the  $n$ -jet of functions  $u : \mathcal{W} \rightarrow \mathbb{R}$ . That is to say,  $\tilde{u}$  is nilomorphic if and only if there is some  $u$  and some  $f_0$  in  $C^\infty(\mathcal{M}, \mathbb{R})$  such that, for all  $[f] \in J^n \mathcal{W}$ :

$$\tilde{u}([f]) = [u \circ f] + [f_0], \text{ where } [\cdot] \text{ stands for “} n\text{-jet of”}.$$

This condition is independent of the choice of a privileged transversal  $\mathcal{T}$ , which amounts to a change of  $[f_0]$ . So through  $\Psi$ , it works on any  $(\mathcal{M}, N)$  with the invariant factors of  $N$  all of the same degree.

*Note: case  $n = 1$ .* In  $J^1 \mathcal{W} = T\mathcal{W} \xrightarrow{\pi} \mathcal{W}$ , this may be viewed as follows. All  $V \in T_{(m,v)} T\mathcal{W}$  is mapped by  $d\pi|_{(m,v)}$  on  $T_m \mathcal{W}$ . The fibre of  $T\mathcal{W}$  being a vector space, its tangent space at  $(m, v)$  identifies with  $T_m \mathcal{W}$ . Through this identification,  $N(V) := d\pi|_{(m,v)}(V)$  may be viewed as an element of  $T_{(m,v)} T\mathcal{W}$ . By construction, this  $N \in \text{End}(TJ^1 \mathcal{W})$  is 2-step nilpotent. We let the reader check it is the same as the  $N$  built above.

*Note: when  $N$  has invariant factors of different degrees, up to  $n+1$ .* The correspondence between nilomorphic functions and some space of jets may be pursued, with the following adaptation. In that case, the (local) quotient  $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$  has a flag of foliations  $\pi(\mathcal{K}^1) \subset \dots \subset \pi(\mathcal{K}^n) \subsetneq \pi(\mathcal{M})$  that we denote by  $\hat{\mathcal{K}}$ . We may then define a space  $J^{\hat{\mathcal{K}}}(\mathcal{M}/\mathcal{I})$  of “ $\hat{\mathcal{K}}$ -jets” of functions from  $\mathbb{R}$  to  $\pi(\mathcal{M})$ , by:  $f \sim g$  if, for each  $a$ ,  $f$  and  $g$  represent the same jet in  $J^a(\pi(\mathcal{M})/\pi(\mathcal{K}^a))$ . In terms of coordinates adapted to  $\hat{\mathcal{K}}$ , a  $\hat{\mathcal{K}}$ -jet  $[f]$  is the data of the coordinates of  $f$  up to the order  $a$ , as soon as they are transverse to  $\mathcal{K}^a$ . To state the correspondence  $\Psi$ , you have to use, on the left side, coordinates adapted to  $\hat{\mathcal{K}}$  and to factor the coordinates spanning each  $\mathcal{K}^{a+1} \setminus \mathcal{K}^a$  by  $\nu^{n-a}$  *i.e.* to use the  $\nu^{n-n(i)} z_i$  introduced in Example 2.10. Then a function  $\tilde{u} : \mathcal{M} \rightarrow \mathbb{R}[\nu]$  is nilomorphic if and only if it is, locally, the “ $\hat{\mathcal{K}}$ -jet” of some function  $u : \pi(\mathcal{M}) \rightarrow \mathbb{R}$ , plus some other fixed  $\hat{\mathcal{K}}$ -jet  $[f_0]$ .

**2.27 Remark** In view of developing a Differential Calculus he calls “simplicial”, on general topological vector spaces, see *e.g.* [3], W. Bertram has studied those jet bundles. With A. Souvay, he developed a differential calculus on them, using truncated polynomial rings  $\mathbb{K}[X]/(X^n)$  with  $\mathbb{K}$  any topological ring, see [4]. I thank W. Bertram for having pointed me out this reference. The obtained formulas are equivalent, in the case where the  $\mathbb{R}[\nu]$ -module is free *i.e.* all the Jordan blocks of  $N$  have the same size, to some of this section, notably to Proposition 2.16. This equivalence is not immediately explicit in the statements of [4] themselves, but you may observe the “radial expansion” given in Theorem 2.8, and its consequences *e.g.* the expansion of a  $\mathbb{K}[X]/(X^n)$ -valued function given on the bottom of p. 14, in the proof of Theorem 3.6, or the expansion given in the proof of Theorem 2.11.

Besides, as non free  $\mathbb{K}[X]/(X^n)$ -modules are direct sums of free ones, Theorem 4.5 of [4] shows that a generalisation of its formulas to non free modules exists.

**2.28 Remark** A noticeable consequence of Proposition 2.16 is that the fact, for a function  $\mathcal{M} \rightarrow \mathbb{R}$ , to be *polynomial along the leaves of  $\mathcal{I}$* , in the form that appears in its statement, makes sense, independently of the chosen  $N$ -integral local coordinates. So, as the datum of a complex structure on a manifold  $\mathcal{M}$  induces a real analytic structure on it, that of a nilpotent structure induces some “polynomial structure” along the leaves of  $\mathcal{I}$ . In the case  $N^2 = 0$ , this structure is of degree one *i.e.* is a flat affine structure. This may be understood naturally, independently of Prop. 2.16. Take any  $U = NV \in T_m \mathcal{I}$ . There is a unique way, modulo  $\ker N$ , to extend  $V$  in a basic vector field along this leaf  $\mathcal{I}_m$  of  $\mathcal{I}$ . This induces a canonical way to extend  $U$  along  $\mathcal{I}_m$  *i.e.*  $\mathcal{I}_m$  is endowed with a flat affine structure, preserved by  $\text{Diff}(\mathcal{M}, N)$ . In other words, this structure is a flat affine connection  $\nabla$  on  $T\mathcal{I} = J^1 \mathcal{I}$ .

In the case where all the Jordan blocks of  $N$  have the same size  $n$ , using the point of view developed in Example 2.26, we see that the  $\text{Diff}(\mathcal{M}, N)$ -invariant structure on the leaves of  $\mathcal{I}$ , providing their “polynomial structure” is a flat connection  $\nabla$  on its bundle  $J^{n-1} \mathcal{I}$ .

### 2.3 The germs of metrics making parallel some self adjoint nilpotent endomorphism

The link between the preceding sections and our subject is given by the following result.

**2.29 Proposition** *Let  $(\mathcal{M}, g)$  be a pseudo-Riemannian manifold admitting a parallel field  $N \in \Gamma(\text{End}(T\mathcal{M}))$  of self adjoint endomorphisms, nilpotent of index  $n$ . Then  $N$  is integrable and  $g$  is pre-nilomorphic for  $N$ , in the sense of 2.25. After 2.4,  $g$  is the real metric associated with the nilomorphic metric  $h := \sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$ .*

*Conversely, suppose that  $h = \sum_{a=0}^{n-1} \nu^a h_a$ , with  $h_a$  real, is some non degenerate symmetric  $\mathbb{R}[\nu]$ -bilinear form on a manifold  $\mathcal{M}$  with a nilpotent structure  $N$ . Set  $g := h_{n-1}$ . Then  $(\mathcal{M}, g)$  is pseudo-Riemannian,  $N$  is self adjoint and parallel on it — and, according to Definition 2.4,  $h$  is the nilomorphic metric associated with  $g$ .*

The proof needs to recall the following result, to our knowledge (through [6]) first proven by [22], then independently by [18]; see also the short recent version [25].

**2.30 Lemma** *A field of nilpotent endomorphisms on  $\mathbb{R}^d$  with constant invariant factors is integrable if and only if its Nijenhuis tensor  $\mathcal{N}_N$  vanishes and the distributions  $\ker N^k$  are involutive (thus integrable) for all  $k$ .*

**Proof of the proposition.** As  $D$  is torsion free,  $DN = 0$  gives that  $\mathcal{N}_N = 0$ , and that the kernels  $\ker N^a$  are involutive distributions. Then Lemma 2.30 ensures that  $N$  is integrable. Let us check that  $g$  satisfies Definition 2.25. (i) is the fact that  $N$  is self adjoint. (ii) Take  $V$  any nilomorphic vector field, we must check that  $(\mathcal{L}_{NV}g)(A, B) = (\mathcal{L}_V g^1)(A, B)$ , where  $g^1 := g(\cdot, N \cdot)$ , for any vectors  $A, B$ . By Proposition 2.20, we may suppose, without loss of generality, that the field  $V$ , together with some fields extending  $A$  and  $B$ , are coordinate vector fields of some integral coordinates system for  $N$ , so that  $V, A, B, NV, NA$  and  $NB$

commute. Then we must check:  $(NV).(g(A, B)) = V.(g(A, NB))$ .

$$\begin{aligned}
(NV).(g(A, B)) &= g(D_{NV}A, B) + g(A, D_{NV}B) \\
&= g(D_A(NV), B) + g(A, D_B(NV)) \quad \text{as } [V, NA] = [V, NB] = 0, \\
&= g(ND_AV, B) + g(A, ND_BV) \quad \text{as } DN = 0, \\
&= g(D_AV, NB) + g(A, ND_BV) \quad \text{as } N^* = N, \\
&= g(D_VA, NB) + g(A, ND_VB) \quad \text{as } [V, A] = [V, B] = 0, \\
&= g(D_VA, NB) + g(A, D_V(NB)) \quad \text{as } DN = 0, \\
&= V.(g(A, NB)).
\end{aligned}$$

For the converse part, first  $g$  is non degenerate: if  $g(V, \cdot) = 0$  then for any  $a$ ,  $h_a(V, \cdot) = h_{n-1}(V, N^{n-1-a} \cdot) = 0$  so  $V = 0$ . Then  $N^* = N$  is immediate. To ensure  $DN = 0$ , it is sufficient to prove that, for any  $N$ -integral coordinate vector fields  $X_i, X_j, X_k$  and any  $a, b, c$  in  $\mathbb{N}$ ,  $g(D_{N^a X_i} N^b X_j, N^c X_k) = g(N^b D_{N^a X_i} X_j, N^c X_k)$ .

$$\begin{aligned}
&2g(D_{N^a X_i} N^b X_j, N^c X_k) \\
&= (N^a X_i).(g(N^b X_j, N^c X_k)) + (N^b X_j).(g(N^a X_i, N^c X_k)) - (N^c X_k).(g(N^a X_i, N^b X_j)) \\
&= X_i.(g(X_j, N^{a+b+c} X_k)) + X_j.(g(X_i, N^{a+b+c} X_k)) - X_k.(g(X_i, N^{a+b+c} X_j)) \\
&\quad \text{as } h \text{ is nilomorphic, so } g \text{ pre-nilomorphic, see 2.25,} \\
&= 2g(D_{X_i} X_j, N^{a+b+c} X_k),
\end{aligned}$$

which gives in particular the wanted equality.  $\square$

We are done: combining Propositions 2.29 and 2.23 provides exactly Theorem 2.31. In the statement, if needed, see Definitions 2.21, 2.4, 2.11, 2.15 and 2.6 for “nilomorphic”, “associated with”, “nilomorphic coordinates”, “adapted function” and “characteristic signatures”.

**2.31 Theorem** *Let  $N$  be a nilpotent structure of nilpotence index  $n$  on  $\mathcal{M}$ . Then  $N$  is self-adjoint and parallel for a pseudo-Riemannian metric  $g$  if and only if  $g$  is the real metric associated with a nilomorphic metric  $h$ .*

*In nilomorphic local coordinates  $(z_i)_{i=1}^D := ((x_i + (\nu y_i)))_{i=1}^D$ ,  $h$  is an  $\mathbb{R}[\nu]$ -valued,  $\mathbb{R}[\nu]$ -bilinear metric of the form:*

$$\begin{aligned}
h &= \sum_{i,j=1}^D h_{i,j} dz_i \otimes dz_j \quad \text{with nilomorphic functions } h_{i,j} = h_{j,i} \text{ applying in} \\
&\quad \nu^{n-\max(n(i), n(j))} \mathbb{R}[\nu], \text{ and } (h_{i,j})_{i,j=1}^D \text{ non degenerate,} \\
&= \sum_{i,j=1}^D \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{h}_{i,j}}{\partial x^\alpha} (\nu y)^\alpha dz_i \otimes dz_j \quad \text{with } \alpha \text{ a multi-index } (\alpha_i)_{i=1}^D \text{ and } \check{h}_{i,j} = \check{h}_{j,i} \\
&\quad \text{adapted functions of the } (x_i)_i \text{ giving} \\
&\quad \text{the properties of } (h_{i,j})_{i,j=1}^D \text{ above.}
\end{aligned}$$

*The characteristic signatures  $(r_a, s_a)_{a=1}^n$  of the couple  $(N, g)$  are those of  $h$ .*

In the theorem, recall that  $h = \sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$ , as stated in Proposition 2.29; in particular,  $g$  is the coefficient of  $\nu^{n-1}$  in its expansion. See also the important Remark 2.36, and a matricial formulation in Remark 2.40.

**2.32 Remark** Set  $\check{h} = \sum_{i,j=1}^D \check{h}_{i,j} dz_i \otimes dz_j$ . Then  $\check{h} = \sum_{a=0}^{n-1} \check{h}_a \nu^a$  where each  $\check{h}_a$  is the value of  $g(\cdot, N^{n-1-a} \cdot)$  along the transversal  $\{(\nu y) = 0\}$  to  $\mathcal{I}$ . This gives more explicitly the link between  $g$  and  $h$ .

Using Notation 2.33, Corollary 2.34 translates Theorem 2.31 into purely real terms.

**2.33 Notation (i)** If  $((\alpha_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$  is a multi-index designed so that:  $y^\alpha := \prod_{i,a} y_{i,a}^{\alpha_{i,a}}$ , then  $x^\alpha$  and  $\mathcal{L}_{X^\alpha}$  denote:

$$x^\alpha := \prod_{i,a} x_i^{\alpha_{i,a}} = \prod_i x_i^{\sum_a \alpha_{i,a}} \quad \text{and} \quad \mathcal{L}_{X^\alpha} := \frac{\partial^{|\alpha|}}{\partial x^\alpha} := \prod_{i,a} \frac{\partial^{\alpha_{i,a}}}{\partial x_i^{\alpha_{i,a}}}.$$

**(ii)** Using point **(i)**, if  $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$  are  $N$ -integral coordinates, if  $\eta$  is a (multi)-linear form on  $\mathcal{U}/\mathcal{I}$  (hence, depending only on the coordinates  $x_i$ ) and if  $b \in \llbracket 0, n-1 \rrbracket$ , we set:

$$\eta^{(b)} := \sum_{\substack{\alpha \text{ such that} \\ \sum_{i,a} a\alpha_{i,a} = b}} \frac{1}{\alpha!} (\mathcal{L}_{X^\alpha} \eta) y^\alpha.$$

**2.34 Corollary** *In the framework of Theorem 2.31,  $\alpha$  being a multi-index  $((\alpha_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ ,  $g$  is a metric defined by (on the right side, we make use of Notation 2.33 **(ii)**):*

$$g(X_i, N^c X_j) = \sum_{b=c}^{n-1} \sum_{\substack{\alpha \text{ such that} \\ \sum_{i,a} a\alpha_{i,a} = n-1-b}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} (\tilde{B}^{b-c}(X_i, X_j)) y^\alpha = \sum_{b=c}^{n-1} (\tilde{B}^{b-c})^{(n-1-b)},$$

each  $B^a$ , for  $a \in \llbracket 0, n-1 \rrbracket$ , being a  $\pi(\mathcal{K}^{n-1-a})$ -basic symmetric bilinear form on  $\pi(\mathcal{U})$ , non degenerate along  $\pi(\mathcal{K}^{n-a})/\pi(\mathcal{K}^{n-1-a})$ , and  $\tilde{B}^*$  denoting  $\pi^* B^*$ . The characteristic signatures  $(r_a, s_a)_{a=1}^n$  of the couple  $(N, g)$  are the signatures of the  $B^{n-a}$  restricted to  $\pi(\ker N^{n-1-a})$ .

**Proof.** By definition, the form  $h$  of Theorem 2.31 is equal to  $\sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$ . So  $g(X_i, N^c X_j)$  is the coefficient of  $\nu^{n-1-c}$  in  $h$ . Defining the  $\tilde{B}^a$  by  $\tilde{h} = \sum_{a=0}^{n-1} \nu^a \tilde{B}^a$ , we let the reader expand  $h$  as given in Theorem 2.31 in powers of  $\nu$ , proving the Corollary.  $\square$

**2.35 Remark**  $B^a$  is the matrix of  $g(\cdot, N^{n-1-a} \cdot) = h_a$  on the transversal  $\{(\nu y) = 0\}$  to  $\mathcal{I}$ .

**2.36 Important Remark** As any parallel field of endomorphisms is integrable, by Lemma 2.30, Theorem 2.31 and Corollary 2.34 give a parametrisation of the set of pseudo-Riemannian metrics on  $\mathcal{U}$ , with a holonomy representation preserving a self adjoint nilpotent endomorphism  $N$  of a given similarity type. The parameters are the  $(B^a)_{a=0}^{n-1}$ , up to an action of  $C^\infty(\mathbb{R}^{d-\text{rk } N}, \mathbb{R}^{\text{rk } N})$ . Indeed, the  $B^a$  are chosen freely, and characterise  $g$  once the level  $\{(\nu y) = 0\}$ , *i.e.* some section of  $\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}$ , is chosen. As  $\dim \mathcal{I} = \text{rk } N$ ,  $C^\infty(\mathbb{R}^{d-\text{rk } N}, \mathbb{R}^{\text{rk } N})$  acts simply transitively on those sections. We do not describe this action here, however we give an idea of it in Remark 2.43. Moreover some part of the  $B^a$  is invariant under it, see Remark 2.37.

**2.37 Remark** The adapted bilinear form  $\tilde{h}$  in Theorem 2.31 depends on the chosen transversal  $\mathcal{T} = \{(\nu y) = 0\}$ ; see a similar remark for nilomorphic functions in 2.19. Yet the restriction of each  $\tilde{h}_a = g(\cdot, N^{n-1-a} \cdot)$  to the leaves of  $\mathcal{K}^{n-a}$ , — encoded below by the matrix  $\tilde{H}_a^0$  in Remark 2.40 and by the matrix  $B_a^0$  in Example 2.39 —, is canonical *i.e.* does not depend on the choice of  $\mathcal{T}$ . Indeed for each  $a$ ,  $g(\cdot, N^a \cdot)$  does not pass on the quotient  $\pi(\mathcal{U}/\mathcal{K}^a)$ , but its restriction to the leaves of  $\pi(\mathcal{K}^{a+1})$  does, see Remark 2.3. This invariant shall be noticed.

Therefore, it is natural to choose coordinates  $(x_i; n(i) = a + 1) = (x_i)_{i=1+D_a}^{D_{a+1}}$  satisfying some property with respect to it, if they exist. For example, if  $h_{n-1-a} = g(\cdot, N^a \cdot)$  is flat along each leaf of  $\pi(\mathcal{K}^{a+1})$ , one can take coordinates such that its matrix  $B_0^{n-1-a}$  is  $I_{r_{a+1}, s_{a+1}}$ . In particular if  $d_{a+1} = 1$  we can take  $B_0^{n-1-a} \equiv \pm 1$ , after the corresponding characteristic signature.

As the  $\check{h}_a$ , except for their restriction to  $\ker N^{n-a}$ , depend on  $\mathcal{T}$ , may  $\mathcal{T}$  be chosen such that they satisfy some specific property ? The answer is given in Remarks 2.41 and 2.43.

As it appears in Corollary 2.34, the general form of such a metric is cumbersome. However in some cases it becomes simpler. We present two of them. The first one **(A)** is when  $\text{Im } N^p = \ker N^{n-p}$  for  $p \leq n$ , so when all Jordan blocks of  $N$  are of size  $n$ :

$$N \text{ is conjugated to } \begin{pmatrix} 0 & I_{d_n} & & \\ & \ddots & \ddots & \\ & & 0 & I_{d_n} \\ & & & 0 \end{pmatrix}, \text{ with } n \text{ null blocks on the diagonal,}$$

$d_n$  being the  $n^{\text{th}}$  characteristic dimension of  $N$ , the other ones being null. In other terms, the  $\mathbb{R}[\nu]$ -module  $E = (\mathbb{R}^d, N) = (\mathbb{R}^{nd_n}, N)$  is free:  $E \simeq \mathbb{R}[\nu]^{d_n}$ . The second one **(B)** is when the nilpotence index of  $N$  is small, namely we took  $N^3 = 0$ .

**2.38 Example A** After Corollary 2.34, a metric  $g$  makes  $N$  self adjoint and parallel if and only if, in  $N$ -integral coordinates giving  $N$  the block form displayed above, its matrix reads:

$$\text{Mat}(g) = \begin{pmatrix} 0 & \cdots & 0 & G^0 \\ \vdots & \ddots & \ddots & G^1 \\ 0 & \ddots & \ddots & \vdots \\ G^0 & G^1 & \cdots & G^{n-1} \end{pmatrix},$$

with the  $G^a$  defined as follows. For each  $a$ , we denote also by  $B^a$  the matrix of the form  $B^a$  introduced in Corollary 2.34. Using Notation 2.33 **(ii)**:

$$G^a = \sum_{b=0}^{n-1-a} (B^{a-b})^{(b)}.$$

The  $B^a$  are symmetric matrices, function of the coordinates  $x_i$ , with  $B^0$  non degenerate. For each  $a$ ,  $B^a$  represents  $g(\cdot, N^{n-1-a} \cdot)$  along  $\{(\nu y) = 0\}$ . The couple  $(N, g)$  has only one characteristic signature, namely  $(r_n, s_n) = \text{sign}(B^0)$ . So  $G^0 = B^0$ ,  $G^1 = B^1 + \sum_i \left(\frac{\partial B^0}{\partial x_i}\right) y_{i,1}$  etc.; as an example, let us expand  $G^3$ :

$$\begin{aligned} G^3 &= B^3 + B^{2(1)} + B^{1(2)} + B^{0(3)} \\ &= B^3 + \sum_i \left(\frac{\partial B^2}{\partial x_i}\right) y_{i,1} \\ &\quad + \frac{1}{2!} \sum_{i,j} \left(\frac{\partial^2 B^1}{\partial x_i \partial x_j}\right) y_{i,1} y_{j,1} + \sum_i \left(\frac{\partial B^1}{\partial x_i}\right) y_{i,2} \\ &\quad + \frac{1}{3!} \sum_{i,j,k} \left(\frac{\partial^3 B^0}{\partial x_i \partial x_j \partial x_k}\right) y_{i,1} y_{j,1} y_{k,1} + \sum_{i,j} \left(\frac{\partial^2 B^0}{\partial x_i \partial x_j}\right) y_{i,2} y_{j,1} + \sum_i \left(\frac{\partial B^0}{\partial x_i}\right) y_{i,3}. \end{aligned}$$

If  $n = 2$ ,  $\text{Mat}(g)$  is an affine function of the  $y_{i,1}$ :

$$\text{Mat}(g) = \begin{pmatrix} 0 & B^0 \\ B^0 & B^1 + \sum_i \frac{\partial B^0}{\partial x_i} y_{i,1} \end{pmatrix}.$$

**2.39 Example B** Recall that:

$$\begin{aligned} d_1 &= \dim(\pi(\ker N)), \quad d_2 = \dim(\pi(\ker N^2)/\pi(\ker N)) \\ \text{and } d_3 &= \dim(\pi(\ker N^3)/\pi(\ker N^2)) = \dim(\pi(T\mathcal{M})/\pi(\ker N^2)), \end{aligned}$$

see Notation 2.1. We order the coordinates as:

$$((y_{i,2})_{i=d_1+d_2+1}^{d_1+d_2+d_3}, (y_{i,1})_{i=d_1+1}^{d_1+d_2+d_3}, (x_i)_{i=1}^{d_1+d_2+d_3}).$$

So, if  $N^3 = 0 \neq N^2$ :

$$\text{Mat}(N) = \begin{pmatrix} 0 & 0 & I_{d_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} \text{with columns and lines of respective sizes} \\ d_3, d_2, d_3, d_1, d_2, d_3. \text{ Column 1 corre-} \\ \text{sponds to the } y_{i,2}, \text{ columns 2 and 3 to} \\ \text{the } y_{i,1} \text{ and columns 4 to 6 to the } x_i. \end{array}$$

Then a metric  $g$  makes  $N$  self adjoint and parallel if and only if its matrix reads:

$$\text{Mat}(g) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & G^0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} G^1 \end{pmatrix} \\ 0 & 0 & G^0 & 0 & \begin{pmatrix} G^1 \end{pmatrix} \\ 0 & 0 & 0 & \begin{pmatrix} G^1 \end{pmatrix} & \begin{pmatrix} G^2 \end{pmatrix} \\ 0 & \begin{pmatrix} G^1 \end{pmatrix} & \begin{pmatrix} G^1 \end{pmatrix} & \begin{pmatrix} G^2 \end{pmatrix} \\ G^0 & \begin{pmatrix} G^1 \end{pmatrix} & \begin{pmatrix} G^1 \end{pmatrix} & \begin{pmatrix} G^2 \end{pmatrix} \end{pmatrix}$$

with, using again Notation 2.33 (ii):

$$G^0 = B^0, \quad G^1 = B^1 + \begin{pmatrix} 0 & 0 \\ 0 & B^{0(1)} \end{pmatrix} \quad \text{and} \quad G^2 = B^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} B^{1(1)} \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B^{0(2)} \end{pmatrix},$$

where the  $B^a$  are symmetric matrices:

$$\begin{aligned} B^0 &\text{ depending on the } (x_i)_{i>d_1+d_2}, \\ B^1 &\text{ on the } (x_i)_{i>d_1} \\ \text{and } B^2 &\text{ on all the } (x_i)_{i=1}^{d_1+d_2+d_3}. \end{aligned}$$

We recall that:

$$B^{a(1)} = \sum_i \left( \frac{\partial B^a}{\partial x_i} \right) y_{i,1}, \quad B^{a(2)} = \frac{1}{2!} \sum_{i,j} \left( \frac{\partial^2 B^a}{\partial x_i \partial x_j} \right) y_{i,1} y_{j,1} + \sum_i \left( \frac{\partial B^a}{\partial x_i} \right) y_{i,2}.$$

To ensure the non degeneracy of  $g$ , we must also exactly require the nondegeneracy condition stated in Corollary 2.34, which amounts here to:  $B^0$  is non degenerate and:

$$B^1 = \begin{pmatrix} B_0^1 & * \\ * & * \end{pmatrix} \text{ and } B^2 = \begin{pmatrix} B_0^2 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \text{ with } B_0^1 \text{ and } B_0^2 \text{ non degenerate.}$$

The characteristic signatures of  $(N, g)$ , cf. Remark 2.3, are:

$$(\text{sign}(B_0^2), \text{sign}(B_0^1), \text{sign}(B_0^0)) = ((r_1, s_1), (r_2, s_2), (r_3, s_3)),$$

$$\text{so: } \text{sign}(g) = (d_2 + d_3 + r_1 + r_3, d_2 + d_3 + s_1 + s_3).$$

If  $N^2 = 0$ , relabelling the  $G^a$  and  $B^a$  we find, setting  $B^1 = \begin{pmatrix} B_0^1 & B^{1'} \\ {}^t B^{1'} & B^{1''} \end{pmatrix}$ :

$$\text{Mat}(N) = \begin{pmatrix} 0 & 0 & I_{d_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ Mat}(g) = \begin{pmatrix} 0 & 0 & G^0 \\ 0 & G^1 & \\ G^0 & G^1 & \end{pmatrix} = \begin{pmatrix} 0 & 0 & B^0 \\ 0 & B_0^1 & B^{1'} \\ B^0 & {}^t B^{1'} & B^{1''} + B^{0(1)} \end{pmatrix},$$

with  $B^0$  and  $B_0^1$  non degenerate,  $B^0$  depending on the  $(x_i)_{i>d_1}$  and  $B^1$  on all the  $(x_i)_{i=1}^{d_1+d_2}$ . We recall that  $B^{0(1)} = \sum_i (\frac{\partial B^0}{\partial x_i}) y_{i,1}$ . In case  $d_1 = 0$  we re-find the end of Example A above.

**2.40 Remark** One may extrapolate from the case  $N^2 \neq N^3 = 0$ , if needed, the computation of  $\text{Mat}(g)$  for any nilpotence index  $n$ , ordering the coordinates as  $((y_{i,n-1})_i, \dots, (y_{i,1})_i, (x_i)_i)$ . The principle remains the same and no new phenomenon appears. However, the matrix becomes rapidly very cumbersome. So it seems that the use of real coordinates, forms and matrices, if it may be avoided, should be, and replaced by the use of  $\mathbb{R}[\nu]$ -linear ones like in Theorem 2.31, just as complex expressions replace real ones in complex geometry.

Matricially, the form  $h$  appearing in Theorem 2.31 reads as follows. Take  $(z_i)_{i=1}^D$  nilomorphic coordinates of  $\mathcal{U}$ , ordered by increasing values of  $n(i)$ . So,  $(z_i)_{i=1}^{D_a} = (z_i; n(i) \leq a)$  parametrise the leaves of  $\mathcal{K}^a$ . Using Proposition 2.5,

$$\text{Mat}(h) = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \check{H} \right) (\nu y)^{\alpha}$$

with  $\check{H}$  a sum of symmetric matrices  $\nu^a \begin{pmatrix} 0 & 0 \\ 0 & \check{H}^a \end{pmatrix} \in \nu^a \text{M}_D(\mathbb{R})$  satisfying, for each  $a \in \llbracket 0, n-1 \rrbracket$ :

(i) (*Adaptation condition, see 2.15*) the block  $\check{H}^a$  corresponds to the coordinates  $(z_i; n(i) \geq n-a)$ , moreover  $\check{H}^a$  depends only on the  $(x_i; n(i) \geq n-a)$ ,

(ii) (*Non degeneracy condition, see 2.5*) cutting the  $(z_i; n(i) \geq n-a)$  into  $((z_i; n(i) = n-a), (z_i; n(i) > n-a))$ ,  $\check{H}^a$  splits into  $\begin{pmatrix} \check{H}_0^a & * \\ * & * \end{pmatrix}$  with  $\check{H}_0^a$  non degenerate.

Finally, notice that  $\check{H}^a = \nu^a B^a$ , with  $B^a = \text{Mat}(g(\cdot, N^{n-1-a} \cdot))$  along  $\{(\nu y) = 0\}$ , as appearing in Corollary 2.34.

**2.41 Remark** An alternative choice of “preferred” coordinates  $(x_i)_{i=D_{a+1}}^{D_{a+1}} = (x_i; n(i) = a+1)$  for some  $a$  could be to try to get  $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & B^{a''} \end{pmatrix}$ , that is to say to choose them so that the  $\{\frac{\partial}{\partial x_i}; n(i) > a+1\}$  be  $B^a$ -orthogonal to  $\mathcal{K}^{a+1} \cap \{(\nu y) = 0\}$ . It is impossible, as the orthogonal distribution to  $\mathcal{K}^{n-a} \cap \{(\nu y) = 0\}$  is not integrable in general, even by seeking an “adequate” transversal  $\mathcal{T} = \{(\nu y) = 0\}$  to  $\mathcal{I}$ . See Remark 2.43 for a proof. Notice here

that on the contrary, this is possible for *alternate* nilomorphic bilinear forms, for which we can even achieve  $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & 0 \end{pmatrix}$ , see Remarks 3.7 and 3.8.

Here, the only small possibility is, if  $\sum_{c=1}^b d_{n-a+c} = 1$  for some  $b \geq c+1$  (all  $d_{n-a+c}$  null except one), to put a null line under  $B_0^a$  and a null column on its right. Indeed in this case, the  $B^a$ -orthogonal to  $\mathcal{K}^{n-a} \cap \mathcal{T}$  is 1-dimensional inside of the leaves of  $\mathcal{K}^{n-a+b} \cap \mathcal{T}$ , so may anyway be integrated, within these leaves. In this case, it follows moreover from [8] that you may choose the transversal  $\mathcal{T}$  so that the unique coordinate vector  $\frac{\partial}{\partial x_i}$  transverse to  $\mathcal{K}^{n-a}$  in  $\mathcal{K}^{a+b} \cap \mathcal{T}$  has a constant  $B^a$ -square, *e.g.* is isotropic. This puts a constant coefficient below  $B_0^a$  on the right of it. See Example 2.42 just below. Notice that, if moreover  $d_{n-a} = 1$ , you must choose between achieving all this, and achieving  $B_0^a \equiv \pm 1$  as in 2.37.

**2.42 Example** Let us apply both preceding remarks to the Lorentzian case. The only indecomposable germs of Lorentzian metric making a non trivial self adjoint endomorphism  $N$  parallel are those with holonomy algebra included in:

$$\left\{ \begin{pmatrix} 0 & L & 0 \\ 0 & A & -{}^t L \\ 0 & 0 & 0 \end{pmatrix}; A \in \mathfrak{so}(n-2), L \in \mathbb{R}^{n-2} \right\}, \text{ in a basis where } g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then  $N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  is parallel. Writing  $g$  on the form given at the end of 2.39 and setting  $B^0 = 1$  by Remark 2.37 and  $B^{1'} = 0$  by Remark 2.41, we get:

$$\text{Mat}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & B_0^1 & 0 \\ 1 & 0 & b^{1''} \end{pmatrix}, \text{ with } B_0^1 \text{ and } b^{1''} \text{ not depending on the first coordinate.}$$

Again by Remark 2.41, one can get  $b^{1''}$  constant (zero or not), getting a classical form for  $g$ .

**2.43 Remark** The adapted bilinear form  $\check{h}$  appearing in Theorem 2.31 depends on the chosen transversal  $\mathcal{T} = \{(\nu y) = 0\}$ . Let us illustrate here how this dependence works through the example where  $N^2 = 0$ . So in  $N$ -adapted coordinates:

$$((y_i)_{i=1}^{d_2}, (x_i)_{i=1}^{d_1}, (x_i)_{i=d_1+1}^{d_2}), \text{ we get: } \text{Mat}(N) = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $\text{Mat}(g)$  is as given at the end of Example 2.39. Then  $B^0$  is the matrix of  $h_0 = g(\cdot, N \cdot)$ ; this metric is well defined on  $\mathcal{M}/\mathcal{K}$ , so does not depend on  $\mathcal{T}$ , and is, along  $\mathcal{T}$ , a degenerate metric;  $B^1$  is the matrix of  $\check{h}_1$ , the value of  $g$  along  $\mathcal{T}$ , applied to vectors of  $T\mathcal{T}$ . It depends on  $\mathcal{T}$ . The manifold  $\mathcal{T}$  is the image of a section of  $\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}$ . Changing  $\mathcal{T}$  amounts to add to this section a vector field  $U$  defined along  $\mathcal{T}$ , and tangent to  $\mathcal{I}$ . Indeed,  $\mathcal{I}$  is endowed with a flat affine connection  $\nabla$ , see Remark 2.28, so identifies with its tangent space. In turn, this field  $U$  is equal to  $NV$  with  $V$  a section of  $\pi(T\mathcal{M})/\pi(\ker N)$  defined on  $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$ . If  $\sum_{i=d_1+1}^{d_2} v_i \frac{\partial}{\partial x_i}$  represents  $V$ , so that  $U = \sum_{i=1}^{d_2} v_i \frac{\partial}{\partial y_i}$ , it follows from the expression of  $\text{Mat}(g)$  that  $B^1$  becomes  $B_V^1$  given by:

$$B_V^1\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = B^1\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) + V \cdot \left(B^0\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\right) + \sum_{i=1}^D \frac{\partial v_i}{\partial x_j} B^0\left(V, \frac{\partial}{\partial x_k}\right) + \frac{\partial v_i}{\partial x_k} B^0\left(\frac{\partial}{\partial x_j}, V\right) \\ \text{i.e. } B_V^1 = B^1 + \mathcal{L}_V B^0.$$



The Lie derivative  $\mathcal{L}_V B^0$  is well defined, even if  $V$  is defined modulo  $\ker N$ , as  $\ker N = \ker B^0$ . So  $\check{h}_1$  is defined, through the choice of  $\mathcal{T}$ , up to addition of an infinitesimal deformation of  $h_0 = g(\cdot, N \cdot)$  by a diffeomorphism.

We see that the orbit  $\{B_V^1, V \in \Gamma(T\pi(\mathcal{M}))\}$  of the matrix  $B^1$  when  $\mathcal{T}$  varies does not contain in general a matrix such that  $B^{1'} = 0$  and/or  $B^{1''} = 0$ , with the notation of Example 2.39. Getting  $B^{1'} = 0$  is impossible in general if  $\dim(\mathcal{K} \cap \mathcal{T})$  is greater than 1. Indeed for all  $i > d_1$  the 1-form  $(\mathcal{L}_V B^0)(X_i, \cdot)$  is closed along the leaves of  $\mathcal{K} \cap \mathcal{T}$ . So  $d(\iota_{X_i} B^1)$  is canonical, independent of  $\mathcal{T}$ . This proves the impossibility announced in Remark 2.41. Getting  $B^{1''} = 0$  is also impossible in general: this is given by what follows.

To simplify, we suppose now that  $d_1 = 0$  *i.e.*  $\ker N = \text{Im } N$ ; now  $h_0$  is a non degenerate metric on  $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$ . After the infinitesimal part of Ebin's slice theorem (see *e.g.* [5] 4.2–5 for a short explanation):

$$\text{Sym}^2 T^* \mathcal{M} = \{\mathcal{L}_V h_0; V \in \Gamma(T\pi(\mathcal{M}))\} \oplus \ker \delta_{h_0}$$

with  $\delta_{h_0}$  the divergence operator with respect to  $h_0$ . I thank S. Gallot for having let me think to this. Consequently there is a local privileged choice of  $\mathcal{T}$ , making  $\check{h}_1$  divergence free; this divergence free  $\check{h}_1$  is therefore canonical; besides it is given by  $\frac{D(D-1)}{2}$  functions of  $D$  variables. So the choice of  $\mathcal{T}$ , which depends on  $D$  functions of  $D$  variables, acts “with a slice” on the value of  $\check{h}_1$ , and each orbit of this action consists of forms depending also on  $D$  functions of  $D$  variables. Finally, up to diffeomorphism,  $g$  depends on  $D(D-1)$  functions of  $D$  variables:  $2\frac{D(D+1)}{2}$  for the couple  $(B^0, B^1)$ , minus the choice of  $\mathcal{T}$  and of a chart of  $\mathcal{M}/\mathcal{I}$  *i.e.* twice  $D$  functions of  $D$  variables.

**2.44 Remark [Indecomposability of the obtained metrics — the particular case where  $d_n = 1$ .]** The commutant  $\text{SO}^0(g)^N$  of  $N$  in  $\text{SO}^0(g)$  acts trivially on some subspace  $E_0 \neq \{0\}$  of  $T_m \mathcal{M}$  if and only if  $d_n = 1$  *i.e.*, denoting by  $n = n_1 \geq n_2 \geq \dots \geq n_D$  the sizes of the Jordan blocks of  $N$ , if and only if  $n_2 < n$ . Then  $E_0 = \text{Im } N^{n_2}$  and there exist  $N$ -adapted coordinates such that the coordinate vector fields  $(Y_{1,a})_{a=n_2}^{n-1}$  span  $E_0$  and are parallel. In such coordinates,  $\text{Mat}(g)$  does not depend on the coordinates  $(y_{1,a})_{a=n_2}^{n-1}$  and, with the notation of Corollary 2.34, the matrices  $(B^0, \dots, B^{n-n_2-1})$ , which are scalars as they are defined on the 1-dimensional quotient  $\pi(\mathcal{M}/\mathcal{K}^{n_2})$ , are constant; we may choose the coordinates to make them equal to  $(\pm 1, 0, \dots, 0)$ , according to the signature of  $g(\cdot N, \cdot)$ .

In particular,  $\text{SO}^0(g)^N$  stabilises a *nondegenerate* subspace if and only if  $n_2 < \frac{n}{2}$ . Namely, it stabilises any complement of  $\ker(g|_{E_0})$  in  $E_0$ , *e.g.*  $\text{span}((Y_{1,a})_{a=n_2}^{n-1-n_2})$ . Then  $(\mathcal{M}, g)$  is a (local) Riemannian product; this is consistent with the properties of  $\text{Mat}(g)$  described above.

As the generic holonomy group of a metric making  $N$  parallel is  $\text{SO}^0(g)^N$ , see Theorem 3.2 (b), the case  $n_2 < \frac{n}{2}$  is the only one where a generic  $g$  is a (local) Riemannian product.

A similar statement holds if  $g$  makes a self adjoint complex structure parallel, with everywhere  $\mathbb{C}$  replacing  $\mathbb{R}$  and complex groups and dimensions replacing real ones. In particular the condition “ $d_n = 1$ ” becomes “ $\dim_{\mathbb{C}}(T\mathcal{M}/\text{Im } N^{n-1}) = 1$ ” *i.e.*  $d_n = 2$ . The pseudo-Riemannian metric  $g$  is the real part of the appearing complex-Riemannian metric.

*Proof.* The first algebraic statement follows from the form of the matrix of the elements of  $\mathfrak{o}(g)^N$ . Indeed, take a basis in which  $\text{Mat}(N) = \text{diag}(N_{n_1}, \dots, N_{n_D})$  and  $\text{Mat}(g) = \text{diag}(\pm K_{n_1}, \dots, \pm K_{n_D})$ , where:

$$N_p := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \text{M}_p(\mathbb{R}) \quad \text{and} \quad K_p := \begin{pmatrix} & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & \end{pmatrix} \in \text{M}_p(\mathbb{R}).$$

This is possible after [17]. Then  $M \in \mathfrak{o}(g)^N$  if and only if the block-decomposition  $(M_{i,j})_{i,j=1}^D$  of its matrix, corresponding to the Jordan blocks of  $N$ , is such that the diagonal  $M_{i,i}$  are null and, for  $i < j$ :

$$\begin{cases} M_{i,j} = \begin{pmatrix} M'_{i,j} & \\ & 0_{n_i-n_j, n_j} \end{pmatrix} \\ M_{j,i} = \begin{pmatrix} 0_{n_j, n_i-n_j} & M'_{i,j} \end{pmatrix} \end{cases} \quad \text{where } M'_{i,j} = \pm \begin{pmatrix} m_{i,j}^1 & m_{i,j}^2 & \dots & m_{i,j}^{n_j} \\ & m_{i,j}^1 & \ddots & \vdots \\ & & \ddots & m_{i,j}^2 \\ & & & m_{i,j}^1 \end{pmatrix} \in M_{n_j}(\mathbb{R}),$$

where  $0_{p,q} = 0 \in M_{p,q}(\mathbb{R})$ , and where the  $\pm$  signs are adequate (we do not take care of them here). Then the reader may check that no non null vector is  $\mathfrak{o}(g)^N$ -stable as soon as  $n_2 = n(=n_1)$ . If  $n_2 < n$ , then  $\mathfrak{o}(g)^N$  acts trivially exactly on the subspace spanned by the  $n - n_2$  first basis vectors *i.e.* on  $\text{Im } N^{n_2}$ . One may then take  $N$ -adapted coordinates such that the coordinate vectors  $(N^a \frac{\partial}{\partial x_1})_{a=n_2}^{n-1} = (Y_{1,a})_{a=n_2}^{n-1}$  span it and are parallel. A way to see it is, for example, the following:

- Take  $N$ -adapted coordinates such that  $N^{n-1} \frac{\partial}{\partial x_1} \neq 0$ ; the metric  $g(\cdot, N^{n-1} \cdot)$  is well-defined on  $\mathcal{M}/\mathcal{K}^{n-1}$ , which is 1-dimensional, so you may modify the coordinate  $x_1$  so that  $g(\frac{\partial}{\partial x_1}, N^{n-1} \frac{\partial}{\partial x_1}) \equiv \pm 1$  *i.e.*  $B^0 = \pm 1$ . If  $n - n_2 = 1$  we are done.

- If  $n - n_2 > 1$ , then  $\pi(\mathcal{M}/\mathcal{K}^{n-2}) = \pi(\mathcal{M}/\mathcal{K}^{n-1})$  is also 1-dimensional so the matrix  $B^1$  is also a scalar. So modifying the level  $\mathcal{T} = \{(\nu y) = 0\}$  as in Remark 2.43 to get:  $B_V^1 = B^1 + \mathcal{L}_V B^0$ , with  $V$  a vector field on  $\mathcal{M}/\mathcal{K}^{n-1}$ , enables to modify arbitrarily  $B^1$ . An adequate  $V$  provides  $B_V^1 \equiv 0$ . The new level  $\{(\nu y) = 0\}$  is obtained by addition of the field  $N^{n-1}V$  *i.e.*, setting  $V = v \cdot \pi(\frac{\partial}{\partial x_1})$ , by the replacement of the coordinate  $y_{1,n-1}$  by  $y_{1,n-1} - v$ .

- We modify then similarly the coordinates  $y_{1,n-a}$ , for  $a \in \llbracket 1, n - n_2 - 1 \rrbracket$ , by induction on  $a$ : at each step, we choose  $V = v \cdot \pi(\frac{\partial}{\partial x_1})$  such that  $B^a + \mathcal{L}_V B^0 \equiv 0$  and replace  $y_{1,n-a}$  by  $y_{1,n-a} - v$ . We let the reader check that this makes the scalars  $B^1, \dots, B^{n-n_2-1}$  null.

It follows then from the form of the metric given by Corollary 2.34 that no coefficient of the metric  $g$  depends on the coordinates  $(y_a)_{a=n_2}^{n-1}$ . At  $m$ , we may take the  $(N^a \frac{\partial}{\partial x_1})_{a=n-1}^0$  such that, on the subspace they span,  $\text{Mat}(N) = N_n$  and  $\text{Mat}(g) = \pm K_n$  with the notation introduced above. Then  $\text{SO}^0(g)^N$  acts trivially exactly on  $\text{Im } N^{n_2} = \text{span}((N^a \frac{\partial}{\partial x_1})_{a=n-1}^{n_2})$ . One sees that this space is totally isotropic except if  $n_2 < \frac{n}{2}$ , where the announced decomposability property arises. The fact that  $\text{SO}^0(g)^N$  preserves no other non degenerate subspace follows from its matricial expression given above.

**2.45 Example** We end with an example of a pseudo-Riemannian metric where a parallel  $N$  naturally arises. I thank L. Bérard Bergery for having told me that this example does not require the metric to be locally symmetric. In §2.2, Example 2.26 showed that jet bundles carry naturally a nilpotent structure; in particular, tangent bundles carry a 2-step nilpotent structure  $N$ , see the first *Note* at the end of Ex. 2.26. Now if  $g$  is a (pseudo-)Riemannian metric on  $\mathcal{M}$  of dimension  $d$ , one of the metrics naturally associated with it on  $T\mathcal{M}$  makes  $N$  self adjoint and parallel. This  $\hat{g}$ , of signature  $(d, d)$ , is defined as follows. In  $TT\mathcal{M}$ , we call “vertical” the tangent space  $\mathbf{V}$  to the fibre, and  $\mathbf{H}$  its “horizontal” complement given by the Levi-Civita connection  $D$  of  $g$ . If  $X \in T_{(m,V)}T\mathcal{M}$ , as the fibre of  $T\mathcal{M}$  is a vector space,  $X$  is naturally identified with a vector of  $T_m\mathcal{M}$ , that we denote by  $X'$ . We set  $\pi : T\mathcal{M} \rightarrow \mathcal{M}$ . We define  $\hat{g}$  by:

$$\mathbf{V} \text{ and } \mathbf{H} \text{ are } \hat{g}\text{-totally isotropic and, if } (X, Y) \in \mathbf{V} \times \mathbf{H}, \hat{g}(X, Y) = g(X', d\pi(Y)).$$

Notice that  $N$  is  $\hat{g}$ -self adjoint. Moreover  $\hat{D}N = 0$ , with  $\hat{D}$  the Levi-Civita connection of  $\hat{g}$ .

*Proof.* We must show that, if  $X, Y, Z$  are vector fields tangent to  $T\mathcal{M}$ ,  $\hat{g}(\hat{D}_X NY, Z) = \hat{g}(\hat{D}_X Y, NZ)$ . Let us show it at an arbitrary point  $m \in T\mathcal{M}$ . At this point, we build a frame field  $\beta$  on  $T\mathcal{M}$  as follows. We take normal coordinate vector fields  $(U_i)_{i=1}^d$  at  $\pi(m)$  on  $\mathcal{M}$ . We lift them horizontally on  $T\mathcal{M}$ , getting a frame field  $\beta_{\mathbf{H}}$  of  $\mathbf{H}$ . Besides we denote by  $\beta_{\mathbf{V}}$  the frame field of  $\mathbf{V}$  such that  $\beta'_{\mathbf{V}} = (U_i)_{i=1}^d$ . In other words,  $\beta_{\mathbf{V}} = N.\beta_{\mathbf{H}}$ . We set  $\beta := (\beta_{\mathbf{V}}, \beta_{\mathbf{H}})$ . In the following,  $X, Y, Z$  denote vectors of  $\beta$ . Notice that, by construction,  $\beta_{\mathbf{V}}$  is constant along each fibre of  $T\mathcal{M}$ , so  $[X, Y] = 0$  if  $X, Y \in \beta_{\mathbf{V}}$ ; besides  $[X, Y] \in \mathbf{V}$  if  $X, Y \in \beta_{\mathbf{H}}$ , as they are lifts of commuting fields on  $\mathcal{M}$ . Finally, by definition of  $\mathbf{H}$ , if  $X \in \mathbf{H}$  and  $Y \in \mathbf{V}$ ,  $[X, Y] \in \mathbf{V}$  and  $[X, Y]' = D_{d\pi(X)}Y'$ . As the  $(U_i)_{i=1}^d$  are normal coordinate vectors at  $\pi(m)$ , for all  $(i, j, k)$ ,  $U_i.(g(U_j, U_k))$  is null at  $\pi(m)$ ; combined with the definition of  $\hat{g}$ , it gives that for any  $X, Y, Z$  in  $\beta$ ,  $X.\hat{g}(Y, Z)$  is also null at  $m$ . Then, if  $X, Y, Z \in \beta$ :

$$\begin{aligned} 2\hat{g}(\hat{D}_X NY, Z) &= X.\hat{g}(NY, Z) + NY.\hat{g}(NY, Z) - Z.\hat{g}(X, NY) \\ &\quad - \hat{g}(X, [NY, Z]) - \hat{g}(NY, [X, Z]) + \hat{g}(Z, [X, NY]) \\ &\stackrel{(\text{at } m)}{=} -\hat{g}(X, [NY, Z]) - \hat{g}(NY, [X, Z]) + \hat{g}(Z, [X, NY]). \end{aligned}$$

If  $Y \in \beta_{\mathbf{V}}$ ,  $NY = 0$  so this vanishes. If  $Z \in \beta_{\mathbf{V}}$  (or if  $X \in \beta_{\mathbf{V}}$ , left to the reader) then:

- as  $NY \in \beta_{\mathbf{V}}$ ,  $[NY, Z] = 0$  so  $\hat{g}(X, [NY, Z]) = 0$ ,
- $[X, Z] \in \mathbf{V}$  so  $\hat{g}(NY, [X, Z]) \in \hat{g}(\mathbf{V}, \mathbf{V}) = \{0\}$ ,
- as  $NY \in \beta_{\mathbf{V}}, [X, NY] \in \mathbf{V}$  so  $\hat{g}(Z, [X, NY]) \in \hat{g}(\mathbf{V}, \mathbf{V}) = \{0\}$ .

So  $(X \in \beta_{\mathbf{V}} \text{ or } Y \in \beta_{\mathbf{V}} \text{ or } Z \in \beta_{\mathbf{V}}) \Rightarrow g|_m(\hat{D}_X NY, Z) = 0$ . If  $X, Y, Z \in \beta_{\mathbf{H}}$ , at  $m$ :

$$\hat{g}(\hat{D}_X NY, Z) = \frac{1}{2} (g(d\pi(X), D_{d\pi(Z)} d\pi(Y)) - 0 + g(d\pi(Z), D_{d\pi(X)} d\pi(Y))) = 0$$

as the  $(U_i)_i$  are normal at  $m$ . Now similarly, at  $m$ ,  $2\hat{g}(\hat{D}_X Y, NZ) = -\hat{g}(X, [Y, NZ]) - \hat{g}(Y, [X, NZ]) + \hat{g}(NZ, [X, Y])$ , which also vanishes. So at any  $m$ ,  $\hat{D}_X NY = N\hat{D}_X Y$  i.e.  $\hat{D}N = 0$ , Q.E.D.

We could build a similar  $\hat{g}$  on any jet bundle  $J^n\mathcal{M}$ , making the  $N$  of Ex. 2.26 parallel.

### 3 Metrics such that $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ has an arbitrary semi-simple part and a non trivial radical

We built in section 1, §1.4, metrics such that the semi-simple part  $\mathfrak{s}$  of  $\text{End}(T\mathcal{M})^{\mathfrak{h}}$  is in each of the eight possibilities listed by Theorem 1.11. Then we built in section 2 metrics admitting an arbitrary self adjoint nilpotent structure as a parallel endomorphism; in particular, this makes in general the radical  $\mathfrak{n}$  of  $\text{End}(T\mathcal{M})^{\mathfrak{h}}$  non trivial, namely generically principal i.e. admitting a unique, arbitrary generator. Mixing here both arguments, we build metrics with  $\mathfrak{s}$  arbitrary in the list of Theorem 1.11, and  $\mathfrak{n}$  generated by some arbitrary self adjoint nilpotent endomorphism  $N$ . We considered only the case where  $N$  is in the commutant of  $\mathfrak{s}$ . It is natural: it means that  $N$  is a complex endomorphism if  $\mathfrak{s} = \langle J \rangle$  induces a complex structure etc., see Remark 3.4. Besides by Proposition 1.8 and as we suppose  $N$  self adjoint, the non commuting case is strongly constrained.

This construction produces metrics the holonomy group of which is the commutant of  $N$  in each of the holonomy groups given in Remark 1.15.

The principle is roughly the following. We repeat the constructions recalled in §1.4, on a manifold  $\mathcal{M}$  with a nilpotent structure  $N$ , replacing everywhere real differentiable or complex holomorphic functions by nilomorphic  $\mathbb{R}[\nu]$ -valued, or nilomorphic+holomorphic  $\mathbb{C}[\nu]$ -valued ones. This gives Theorem 3.2. To state it, if  $J$  (or  $L$ ) and  $N$  are (para)complex and nilpotent structures that commute, we need integral coordinates for both  $J$  (or  $L$ ) and  $N$ . They exist, this is the following lemma, proven on p. 37.

**3.1 Lemma** (small enhancement of Lemma 2.30) Suppose that  $\mathbb{R}^{2d}$  is endowed with a (para)complex structure  $J$  (or  $L$ ), and that some nilpotent field  $N$  commutes with it. Coordinates are called here adapted to  $J$  (or  $L$ ) if they make  $\text{Mat}(J)$  (or  $\text{Mat}(L)$ ) constant  $J_1$ - (or  $L_1$ -) block diagonal. Then there is a coordinate system simultaneously adapted to  $J$  (or  $L$ ) and integral for  $N$  if and only if  $N_N = 0$  and the  $\ker N^k$  are involutive for all  $k$ . This holds also with complex coordinates if all those endomorphisms are  $\mathbb{C}$ -linear on  $\mathbb{C}^{2d}$ .

**3.2 Theorem (Corollary and generalisation of Theorem 2.31)** Let  $H_{\mathfrak{s}}$  be the generic holonomy group corresponding to an algebra  $\mathfrak{s}$  in any of the eight cases of Theorem 1.11, and  $N$  any self adjoint nilpotent endomorphism in the commutant of  $\mathfrak{s}$  i.e. the bicommutant of  $H_{\mathfrak{s}}$ . We denote by  $\mathcal{G}_{H_{\mathfrak{s}}^N}$  the set of germs of metrics the holonomy group  $H$  of which is included in the commutant  $H_{\mathfrak{s}}^N$  of  $N$  in  $H_{\mathfrak{s}}$ . If  $g \in \mathcal{G}_{H_{\mathfrak{s}}^N}$ , then  $N$  extends as a parallel endomorphism, in particular as a nilpotent structure. Besides  $\text{End}(T\mathcal{M})^{\mathfrak{h}} \supset \mathfrak{s} \times \langle N \rangle$ .

(a) For  $\mathfrak{s}$  in each case of Theorem 1.11,  $g \in \mathcal{G}_{H_{\mathfrak{s}}^N}$  if and only if:

– In case **(1 $\mathbb{C}$ )**,  $g$  is the real metric associated with the real part of a non degenerate,  $\mathbb{C}[\nu]$ -valued,  $\mathbb{C}[\nu]$ -bilinear metric  $h$ , which is both holomorphic and nilomorphic on  $(\mathcal{M}, \underline{J}, N)$ .

– In case **(2)**, and in  $N$ -adapted coordinates  $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$  such that  $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$  for any odd  $i$ , given by Lemma 3.1,  $g$  is the real metric associated with a non degenerate nilomorphic metric  $h$  given by:

$$\begin{aligned} h\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j}\right) &= \frac{\partial^2 u}{\partial w_i \partial \bar{w}_j}, \text{ with } u \text{ some } \mathbb{R}[\nu]\text{-valued nilomorphic function,} \\ &\text{and } \frac{\partial}{\partial w_{(j+1)/2}} := \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+1}} \in T^{1,0}\mathbb{R}^d \text{ for all odd } j, \\ &= \sum_{\alpha} \frac{\partial^{|\alpha|}}{\partial x_{\alpha}} \left( \frac{\partial^2 \tilde{u}}{\partial w_i \partial \bar{w}_j} \right) (\nu y)^{\alpha} \text{ with } \tilde{u} \text{ some } \mathbb{R}[\nu]\text{-valued adapted} \\ &\text{function of the coordinates } (x_i)_i. \end{aligned}$$

– In case **(2')**, and in  $N$ -adapted coordinates such that  $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$  for any odd  $i$ , given by Lemma 3.1,  $g$  is the real metric associated with a non degenerate nilomorphic metric  $h$  given by:

$$\begin{aligned} h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{j+1}}\right) &= \frac{\partial^2 u}{\partial x_i \partial x_{j+1}} \text{ for } i, j \text{ odd, with } u \text{ some } \mathbb{R}[\nu]\text{-valued nilomorphic function,} \\ &= \sum_{\alpha} \frac{\partial^{|\alpha|}}{\partial x_{\alpha}} \left( \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_{j+1}} \right) (\nu y)^{\alpha} \text{ with } \tilde{u} \text{ some } \mathbb{R}[\nu]\text{-valued adapted} \\ &\text{function of the coordinates } (x_i)_i. \end{aligned}$$

– In case **(2 $\mathbb{C}$ )**, and in complex  $N$ -adapted coordinates such that  $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ , or  $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ , for  $i$  odd,  $g$  is the real part of the complex metric associated with a non degenerate complex nilomorphic metric  $h$  given by the complexifications, which are equivalent, of any of the two formulas given in the previous cases. The potential  $u$  is then a  $\mathbb{C}[\nu]$ -valued holomorphic and nilomorphic function.

– In cases **(3)**, **(3')** and **(3<sup>ℂ</sup>)**, and in the real analytic category, it is the solution of the exterior differential system given in §1.4, formulated in nilomorphic coordinates and with  $\mathbb{R}[\nu]$ - or  $\mathbb{C}[\nu]$ -valued nilomorphic functions instead of real or complex ones. In particular, in cases **(3)** and **(3')**, the elements of  $\mathcal{G}_{H_s^N}$ , considered up to diffeomorphism, are parametrised by  $b\frac{D}{2}$  real analytic functions of  $\frac{D}{2} + 1$  variables, where  $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$ , with  $D$  the number of invariant factors of  $N$  and  $d_a$  the number of repetitions of  $X^a$  among them. In case **(3<sup>ℂ</sup>)**, they are by  $b\frac{D}{4}$  holomorphic functions of  $\frac{D}{4} + 1$  complex variables.

**(b)** In a dense open subset for the  $C^2$  topology in  $\mathcal{G}_{H_s^N}$  (considered in the real analytic category in cases **(3)**, **(3')** and **(3<sup>ℂ</sup>)**), the holonomy group  $H$  of the metric is exactly  $H_s^N$ . In particular, those commutants are holonomy groups. Then also,  $\text{End}(T\mathcal{M})^h = \mathfrak{s} \times \langle N \rangle$ .

**3.3 Remark** For the meaning of  $D$  and the  $d_a$ , see also Notation 2.1.

**3.4 Remark** A nilpotent endomorphism  $N$  commutes with  $\mathfrak{s}$  if and only if  $\text{Id} + N$  does. Now by definition, the automorphisms commuting with  $\mathfrak{s}$  are those preserving the  $G$ -structure defined by the commutant of  $\mathfrak{s}$ . In cases **(1<sup>ℂ</sup>)** and **(2)**, they are exactly the complex automorphisms, with respect to  $\underline{J}$ , respectively  $J$ . In case **(2')**, they are the paracomplex automorphisms *i.e.* those of matrix  $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$  with  $U, V \in \text{GL}_{d/2}(\mathbb{R})$ . In case **(3)**, they are the quaternionic automorphisms, with respect to  $(J_1, J_2, J_3)$ . In case **(3')**, they are the automorphisms with matrix  $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$  with  $U \in \text{GL}_{d/2}(\mathbb{R})$ , if  $\text{Mat}(L) = I_{d/2, d/2}$  and  $\text{Mat}(J) = J_{d/2}$ . Cases **(2<sup>ℂ</sup>)** and **(3<sup>ℂ</sup>)** are the complexification of cases **(2<sup>(ℓ)</sup>)** and **(3<sup>(ℓ)</sup>)**.

**Proof of Lemma 3.1.** We do it in the real case. The “only if” is immediate. The converse is immediate in the case of  $J$ : repeat the proof of [25] with the field  $\mathbb{C}$  replacing  $\mathbb{R}$ :  $\mathbb{R}^{2d}$  has a complex structure and may be considered as  $\mathbb{C}^d$ , then work with complex coordinates. For the case of  $L$ , or of  $J$  for an alternative proof, we have to check that the proof works with  $L$ -adapted coordinates at each step (on p. 610 of [25]). The author builds the coordinates by induction on the nilpotence index  $n$  of  $N$ . For  $n = 1$  the result is empty hence true. If it holds for index  $n - 1$  and if  $N^{n-1} \neq N^n = 0$ , as  $\ker N$  is  $L$ -invariant we may find coordinates  $((x_i)_i, (y_i)_i)$  that are:

- $L$ -adapted *i.e.* such that  $L\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$  and  $L\frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_{i+1}}$  for any odd  $i$ ,
- and such that the  $(y_i)_i$  parametrise the leaves of  $\mathcal{K}$ .

Then the induction assumption applies on  $\mathbb{R}^{2d}/\mathcal{K}$ , providing coordinates  $(x_i)_i$  of the wished type on  $\mathbb{R}^{2d}/\mathcal{K}$ . As the fields  $\left(N\frac{\partial}{\partial x_i}\right)_i$  commute with each other, [25] extends these coordinates to the whole  $\mathbb{R}^{2d}$ , obtaining  $N$ -adapted coordinates  $((x_i)_i, (\bar{y}_i)_i)$ . Here, we need moreover to check that they are also  $L$ -adapted *i.e.* **(i)**  $L\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{i+1}}$  and **(ii)**  $L\left(\frac{\partial}{\partial \bar{y}_i}\right) = \frac{\partial}{\partial \bar{y}_{i+1}}$  for  $i$  odd. We follow [25]: the  $\frac{\partial}{\partial x_i}$  are equal to some  $N\left(\frac{\partial}{\partial x_j}\right)$  if they are in  $\text{Im } N$ , else they are equal to  $\frac{\partial}{\partial x_i}$ . As  $LN = NL$ , this gives **(i)**. The  $\frac{\partial}{\partial \bar{y}_i}$  are equal to some  $N\left(\frac{\partial}{\partial x_k}\right)$  if they are in  $\text{Im } N$ , else they are chosen freely. As  $LN = NL$ , this gives **(ii)**.  $\square$

**3.5 Remark** (*This will be used in Part 4*) The key properties used in the proof are that  $J$ , or  $L$ , has a constant matrix in the basis  $\left(\frac{\partial}{\partial x_i}\right)_i$ , and that it commutes with  $N$ . So the same proof, and result, hold for any integrable field of endomorphism playing the role of  $J$  or  $L$ .

To show the theorem we need the following remarks.

**3.6 Remark (natural matricial form for an  $\mathbb{R}[\nu]$ -bilinear alternate 2-form)** We use here the notation introduced in 2.1 and 2.2 and re-state briefly the relevant adaptation of 2.3, 2.4 and 2.5 for  $\omega$  an  $\mathbb{R}[\nu]$ -bilinear *alternate* 2-form on  $E = (\mathbb{R}^d, N)$ . So:

$$\omega = \sum_{a=0}^{n-1} \nu^a \omega_a \text{ with } \omega_{n-1} \text{ such that } \omega_{n-1}(\cdot, N \cdot) = \omega_{n-1}(N \cdot, \cdot)$$

$$\text{and for any } a, \omega_a = \omega_{n-1}(\cdot, N^{n-1-a} \cdot).$$

For  $a \in \llbracket 1, n \rrbracket$  we denote by  $r_a \in 2\mathbb{N}$  the rank of  $\omega_{n-a} := \omega_{n-1}(\cdot, N^{a-1} \cdot)$  defined on  $\ker N^a / \ker N^{a-1}$ . It is standard that the couple  $(N, \omega_{n-1})$  on  $\mathbb{R}^d$  is characterised up to conjugation by the  $(r_a)_{a=1}^n$ , called here the *ranks* of  $\omega$ . See *e.g.* [17] like in 2.3.

If  $\beta = (X_i)_{i=1}^D$  is an adapted spanning family (see 2.2) of  $E$ ,  $\text{Mat}_\beta(\omega) = \sum_{a=0}^{n-1} \nu^a \Omega_a \in \text{M}_D(\mathbb{R}[\nu])$  where:

(i)  $\Omega_a = \begin{pmatrix} 0 & 0 \\ 0 & \check{\Omega}^a \end{pmatrix}$ , the upper left null square block, of size  $D_{n-1-a}$ , corresponding to  $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a} X_i = 0\}$ ,

(ii) the upper left block  $\check{\Omega}_0^a$  of  $\check{\Omega}^a$  of size  $d_{n-a}$ , corresponding to  $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a} X_i \neq N^{n-a} X_i = 0\}$  is of rank  $r_{n-a}$ . So if  $S \oplus \text{Im } N = E$ ,  $r_a$  is the rank of the (well defined) form  $\omega_{n-a}$  on the quotient  $(S \cap \ker N^a) / (S \cap \ker N^{a-1})$ .

For  $r \leq \delta$  and  $r$  even, we denote by  $J_{\delta, r/2}$  the matrix  $\text{diag}(J_{r/2}, 0) \in \text{M}_\delta(\mathbb{R}[\nu])$ . There are adapted spanning families  $\beta = (X_i)_{i=1}^D$  of  $E$  such that  $\check{\Omega}_a$  is null except  $\check{\Omega}_0^a = J_{d_a, r_a}$ , for all  $a$  *i.e.*:

$$\text{Mat}_\beta(\omega) = \text{diag}(\nu^{n-a} J_{d_a, r_a/2})_{a=1}^n = \begin{pmatrix} \nu^{n-1} J_{d_1, r_1/2} & & \\ & \ddots & \\ & & \nu^0 J_{d_n, r_n/2} \end{pmatrix} \in \text{M}_D(\mathbb{R}[\nu]).$$

Each block  $\nu^{n-a} J_{d_a, r_a/2}$  corresponds to the factor  $(\nu^{n-a} \mathbb{R}[\nu])^{d_a} = \text{span}((X_i)_{D_{a-1} < i \leq D_a})$  of  $E$ . The form  $\omega$  is non degenerate if and only if each  $\omega_a$  is *i.e.*  $r_a = d_a$  for all  $a$ .

**3.7 Remark** The Poincaré and Darboux lemmas admit a natural “nilomorphic” version. For example, if  $B$  is some ball in  $(\mathbb{R}^d, N)$  with  $N$  nilpotent, in constant Jordan form:

(a) If  $\lambda \in \Lambda_{\mathbb{R}[\nu]}^k(B)$  with  $p > 0$  is a closed nilomorphic  $k$ -form on  $B$  then there is a nilomorphic  $\alpha \in \Lambda_{\mathbb{R}[\nu]}^{k-1}(B)$  such that  $\lambda = d\alpha$ .

(b) If  $\omega \in \Lambda_{\mathbb{R}[\nu]}^2(B)$  is closed and has constant ranks, there exist nilomorphic coordinates on  $B$  in which  $\text{Mat}(\omega)$  has the (constant) form given at the end of Remark 3.6.

**Proof.** (a) Classically, if  $\lambda$  is a closed real form and  $X$  a vector field on  $B$  the flow  $(\varphi^t)_{t \in [0, +\infty[}$  of which is a retraction of  $B$  on a point, then:

$$\alpha := \int_0^\infty \iota_X (\varphi^{t*} \lambda) dt$$

fits. In our case, use the retraction  $(e^{-t} \text{Id}_B)_{t \in [0, +\infty[}$  generated by  $X = -\text{Id}_{\mathbb{R}^d}$ ;  $\varphi^t$  and  $X$  being nilomorphic, so is the obtained integral form  $\alpha$ .

(b) As  $\omega$  is closed and has constant ranks, its kernel integrates in some foliation  $\mathcal{F}$  and the action of  $N$  passes to the quotient  $B/\mathcal{F}$ . So we may suppose that  $\omega$  is non degenerate. Then we use Moser's path method. We set  $\omega_0$  the constant nondegenerate 2-form on  $\mathbb{R}^d$ , given in Remark 3.6. To simplify, we suppose that  $\omega_t := \omega_0 + t(\omega - \omega_0)$  never degenerates. Else, iterate the method along an adequate piecewise affine path from  $\omega_0$  to  $\omega$ . We want to build a *nilomorphic* homotopy  $(\varphi^t)_{t \in [0,1]}$  such that:

$$\varphi_t^* \omega_t = \omega_0. \quad (*)$$

Once this is done,  $\varphi_1^* \omega = \omega_0$  and  $\varphi_1^* N = N$  as we want. Let  $X_t$  be the field such that  $X_t(\varphi^t(p)) = \frac{d}{dt} \varphi^t(p)$ , then  $(*)$  amounts to:

$$\varphi^{t*} \left( \mathcal{L}_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) = 0$$

*i.e.*, as  $d\omega_t = 0$ ,  $d(\iota_{X_t} \omega_t) + \omega = 0$ . By (a), there is a *nilomorphic* 1-form  $\lambda$  on  $B$  with  $\omega = d\lambda$ . Then it is sufficient to find  $X_t$  with  $\iota_{X_t} \omega_t + \lambda = 0$ . As  $\omega_t$  is nondegenerate, this defines indeed  $X_t$ ; as  $\omega_t$  and  $\lambda$  are nilomorphic, so is  $X_t$ .  $\square$

**3.8 Remark** An infinitesimal deformation of nilomorphic Darboux coordinates for a non degenerate  $\omega$  is given by a nilomorphic field  $X$  such that  $0 = \mathcal{L}_X \omega = d(\iota_X \omega)$ . So, classically,  $X$  is the symplectic gradient of some (nilomorphic) potential. After Proposition 2.23, such a potential is given by an adapted function  $f = \sum_a f_a \nu^a$  on  $\mathcal{T} = \{(\nu y) = 0\}$  *i.e.* by one real function  $(f_{n-1})$  of  $D - D_0 = D$  variables, one  $(f_{n-2})$  of  $D - D_1$  variables,  $\dots$ , and one  $(f_0)$  of  $D - D_{n-1}$  variables. Set  $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$ . Then  $0 = D_0 = \dots = D_{b-1}$ . So  $f_{n-1}$  up to  $f_{n-b}$  depend, each, on  $D - D_0 = D$  variables — strictly more variables than any other  $f_c$ . So the Darboux coordinates depend on  $b$  functions of  $D$  variables.

In passing, we add the following. After Rem. 2.3, on each leaf of the (quotient) foliation  $\pi(\mathcal{K}^{a+1})/\pi(\mathcal{K}^a)$ , the (real) symplectic form  $\omega(\cdot, N^a \cdot)$  is well defined. The choice of nilomorphic Darboux coordinates means in particular the choice of Darboux coordinates for them, but means also the choice of a good transversal  $\mathcal{T} = \{(\nu y) = 0\}$  to  $\mathcal{I}$  *i.e.* such that for all  $a$ :

- the orthogonal distribution to  $\mathcal{T} \cap \mathcal{K}^a$  with respect to  $\omega(\cdot, N^{a-1} \cdot)$  is integrable,
- its intersection with  $\mathcal{T}$  is totally isotropic.

In fact, it amounts exactly to both these choices.

**Proof of Theorem 3.2.** The fact that  $N$  extends as a nilpotent structure was given by Lemma 2.30 in §2.3. To prove (a) and (b), we begin with cases (3) and (3') — case (3<sup>C</sup>) is only their complexification.

**Part (a)** We follow the line of §1.4. The triple  $(J, U, N)$  is given and we look for a quadruple  $(g, J, U, N)$ . This is equivalent to a quadruple  $(J, N, \tilde{\omega}_0, \tilde{\omega})$  where:

$$\tilde{\omega}_0 := \sum_{a=0}^{n-1} \omega_0(\cdot, N^{n-1-a} \cdot) \nu^a$$

is the  $(J)$ -complex nilomorphic symplectic 2-form associated with the complex symplectic 2-form  $\omega_0 := g(\cdot, U \cdot) + ig(\cdot, JU \cdot)$ , and where:

$$\tilde{\omega} := \sum_{a=0}^{n-1} \omega(\cdot, N^{n-1-a} \cdot) \nu^a$$

is the symplectic nilomorphic (1,1)-form associated with  $\omega := \omega_0(\cdot, U \cdot)$ . As  $UN = NU$ ,  $\tilde{\omega} = \tilde{\omega}_0(\cdot, U \cdot)$ . By Remark 3.7 **(b)**, we may use local coordinates  $(x_j, (y_{j,a})_a)_{j=1}^D$  adapted to  $N$  and such that for all odd  $j$ ,  $J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_{j+1}}$ . As usual, we set  $z_j := x_j + (\nu y_j) \in \mathbb{R}[\nu]$  for all  $j$  and introduce then the “complex and nilomorphic” local coordinates  $(w_j)_{j=1}^{D/2}$  by:

$$w_{(j+1)/2} := z_j + iz_{j+1} \in \mathbb{C}[\nu].$$

Through those coordinates, we consider that we are in some ball  $\mathcal{B}$  of  $(\mathbb{C}^{d/2}, N)$  with  $N$  a complex endomorphism field. The matrix  $\Omega_0$  of  $\omega_0$  as a  $\mathbb{C}[\nu]$ -bilinear form is as at the end of Remark 3.6, with non degenerate  $J_{d_a, r_a/2} = J_{d_a}$ . In the complexification  $T^{\mathbb{C}}\mathcal{B} = T\mathcal{B} \otimes \mathbb{C}$  of the tangent bundle, we introduce also, for  $j$  odd, the vector fields  $\frac{\partial}{\partial w_{(j+1)/2}} := \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+1}}$  and  $\frac{\partial}{\partial \bar{w}_{(j+1)/2}} := \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial x_{j+1}}$ . We introduce the matrix  $V = (v_{i,j})_{i,j=1}^{D/2} \in M_{D/2}(\mathbb{C}[\nu])$  of the  $\nu$ -linear,  $J$ -antilinear morphism  $U$  by:

$$U \left( \frac{\partial}{\partial w_i} \right) = \sum_{j=1}^{D/2} v_{i,j} \frac{\partial}{\partial \bar{w}_j} \quad \text{i.e., if } v_{i,j} = \sum_{a=0}^{n-1} v_{i,j,a} \nu^a, \text{ then:}$$

$$\forall b \in \llbracket 0, n-1 \rrbracket, U \left( N^b \frac{\partial}{\partial w_i} \right) = \sum_{j,a} v_{i,j,a} N^{b+a} \frac{\partial}{\partial \bar{w}_j}.$$

Notice that, as  $N^a \frac{\partial}{\partial \bar{w}_j} = 0$  if  $a \geq n(j)$ ,  $\deg_{\nu} v_{i,j} < n(j)$  for all  $i, j$ . In other terms, in the block decomposition of  $V$  corresponding to the flag of the  $\pi(\ker N^a)$ , i.e. to the blocks of  $\text{Mat}(\omega)$  given in Remark 3.6, the  $\nu$ -degree of the  $a^{\text{th}}$  line of blocks is strictly less than  $a$ .

Let us keep following §1.4. We introduce  $dw$  the column  $(dw_j)_{j=1}^{D/2}$ , then:

$$\omega_0 = {}^t dw \wedge \Omega_0 \wedge dw \quad \text{and, setting } H := -\Omega_0 V, \quad \omega = \frac{i}{2} {}^t dw \wedge H \wedge d\bar{w}.$$

Here, notice that  $\Phi : V \mapsto -\Omega_0 V = H$  is injective. Indeed, it associates with the matrix  $V$  of  $U$ , the matrix  $H$  of the form  $\omega_0(\cdot, U \cdot)$ , and  $\omega_0$  is non degenerate. An alternative, computational argument is the fact that, as  $\deg_{\nu} v_{i,j} < n(j)$  for all coefficient  $v_{i,j}$  of  $V$ , no product  $\nu^a \nu^b$  with  $a + b \geq n$  appears when computing  $\Omega_0 V$ . This shows also the following. Introduce:

$$\Omega_0^{-1} := -\text{diag}(\nu^{-n+1} J_{d_1/2}, \nu^{-n+2} J_{d_2/2}, \dots, \nu^{-1} J_{d_{n-1}/2}, J_{d_n/2}),$$

$\nu^{-a}$  standing for the application  $\sum_{b \geq a} f_b \nu^b \mapsto \sum_{b \geq a} f_b \nu^{b-a}$ , from  $\nu^a \mathbb{C}[\nu]$  to  $\mathbb{C}[\nu]$  i.e. for the left inverse of the multiplication by  $\nu^a$ . Then  $H \mapsto -\Omega_0^{-1} H$  is well-defined on the space of the matrices of  $\mathbb{R}[\nu]$ -bilinear forms, as they satisfy property **(i)** of Remark 3.6. More precisely,  $\Phi$  is a bijection:

$$\Phi : \{V = (v_{i,j}) \in M_D(\mathbb{R}[\nu]); \deg_{\nu} v_{i,j} < n(j) \text{ for all } i, j, {}^t \bar{V} \Omega_0 = -\Omega_0 V \text{ and } V \bar{V} = \varepsilon I\} \rightarrow$$

$$\tilde{\mathcal{H}}_{\varepsilon} := \{H \in M_D(\mathbb{R}[\nu]); H \text{ satisfy property (i) of 3.6, } {}^t \bar{H} = H \text{ and } H \Omega_0^{-1} \bar{H} = \varepsilon \Omega_0\}.$$

Notice that in the set above,  $H$  is necessarily non degenerate, so satisfies property **(ii)** of 3.6 with  $r_a = d_a$  for all  $a$ . Now that this adaptation to the nilomorphic case is done, we may perform Cartan’s test. In fact, it works just like in §1.4. Indeed:

– We look for an  $N$ -stable integral manifold of the exterior differential equation **I** :  ${}^t dw \wedge dH \wedge d\bar{w} = 0$  i.e. for a nilomorphic function  $H : (\mathbb{C}^{d/2}, N) \rightarrow \tilde{\mathcal{H}}_{\varepsilon}$  around the origin,



the graph of which is an integral manifold of  $\mathbf{I}$ . By Proposition 2.16, it amounts to find an *adapted* function  $H : \mathcal{T} \rightarrow \widetilde{\mathcal{H}}_\varepsilon$  (see 2.15), with  $\mathcal{T}$  the transversal  $\{(\nu y) = 0\}$  to the foliation  $\mathcal{I}$ . So in the following we work on  $\mathcal{T}$ , identified with  $\mathbb{C}^{D/2}$  by the coordinates.

– Then, Cartan’s test rests on the equation of the tangent space  $\widetilde{W}_\varepsilon := T_{H(0)}\widetilde{\mathcal{H}}_\varepsilon$ , which we will see to be nearly the same as in §1.4. More precisely, along  $\mathcal{T}$ , the  $M_D(\mathbb{C}[\nu])$ -valued function  $H$  we look for reads  $H = \sum_{a=0}^{n-1} H_a \nu^a$ , each  $H_a$  being the pull back of some complex valued matrix function on  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$ . We will see that the coefficient of  $\nu^a$  in  $\mathbf{I}$  is an exterior differential equation involving only the function  $H_a$ , thus is an exterior differential equation defined on  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$ . Each of those equations is like that of §1.4

To alleviate the formulas, we write them in the case  $\varepsilon = -1$  and  $g$  positive definite. The other cases work alike. We introduce  $\widehat{\Omega}_0 := \text{diag}(J_{d_1/2}, J_{d_2/2}, \dots, J_{d_n/2})$ . As in §1.4, at the origin, we may take  $V = \widehat{\Omega}_0$  *i.e.*, at the origin:

$$H = \widehat{I} := \text{diag}(\nu^{n-1} I_{d_1/2}, \nu^{n-2} I_{d_2/2}, \dots, \nu I_{d_{n-1}/2}, I_{d_n/2}).$$

Then:

$$\begin{aligned} W \in \widetilde{W}_\varepsilon &\Leftrightarrow W \Omega_0^{-1} \widehat{I} + \widehat{I} \Omega_0^{-1} W = 0 \\ &\Leftrightarrow W \widehat{\Omega}_0 + \widehat{\Omega}_0 W = 0 \\ &\Leftrightarrow \sum_{a=0}^{n-1} \nu^a (W_a \widehat{\Omega}_0 + \widehat{\Omega}_0 W_a) = 0. \end{aligned}$$

The coefficient of  $\nu^a$  involves only  $W_a$ . So, as announced, the coefficient of  $\nu^a$  in  $\mathbf{I}$  is an equation involving only  $H_a$ . This equation is stated on the  $(D - D_{n-1-a})/2$ -dimensional quotient  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$  as we look for a  $\mathcal{K}^{n-1-a}$ -basic function  $H_a$ . (This is consistent with the fact that only the bottom right square of  $W_a$ , appearing in (i) of Remark 3.6 and corresponding to the quotient by  $\ker N^{n-1-a}$ , is non vanishing.) Now on this quotient, the vectors may be reordered so that  $\widehat{\Omega}_0$  reads  $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ , and  $W_a \widehat{\Omega}_0 + \widehat{\Omega}_0 W_a = 0$  is the same equation as that defining  $W_\varepsilon$  in §1.4. So Cartan’s criterion is fulfilled and the solutions  $H_a$  depend on  $(D - D_{n-1-a})/2$  of  $(D - D_{n-1-a})/2 + 1$  variables. Finally, set  $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$ . Then  $0 = D_0 = \dots = D_{b-1}$ . So  $H_{n-1}$  up to  $H_{n-b}$  depend, each, on  $(D - D_0)/2 = \frac{D}{2}$  functions of  $(D - D_0)/2 + 1 = \frac{D}{2} + 1$  variables —strictly more variables than any other  $H_c$ . So the whole function  $H$  depends on  $b \frac{D}{2}$  functions of  $(D - D_0)/2 + 1 = \frac{D}{2} + 1$  variables. By Remark 3.8, the choice of the complex Darboux coordinates for  $\omega$  amounts to the choice of  $b$  function of  $D/2$  variables, so this does not interfere.

*Note.* The above technique may be used as a standard reasoning to adapt arguments about an exterior differential system to the nilomorphic framework.

**Part (b) for cases (3), (3’), (3<sup>C</sup>).** Following the note just above, (b) is given by the reasoning referred to in Remark 1.34, adapted as above to the nilomorphic framework, *i.e.* applied to adapted functions, or jets, defined on  $\mathcal{T}$ .

**Part (a) for case (1<sup>C</sup>).** Use the first point of Reminder 1.24, and repeat the very proof of Theorem 2.31, with the field  $\mathbb{C}$  and holomorphic functions replacing  $\mathbb{R}$  and smooth functions.

**Part (a) for cases (2) and (2’), hence (2<sup>C</sup>),** which is their complexification. We are now quicker. Repeat the classic proofs (see respectively *e.g.* [20] §11.2 and §8.3, and [2] §2) with nilomorphic coordinates and functions replacing real ones. In other words:

– Take a nilomorphic coordinate system which is also integral for  $J$  or  $L$  *i.e.* such that  $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$  or  $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$  for all odd  $i$ . This is given by Lemma 3.1.

– Set  $\mathcal{T} := \{(\nu y) = 0\} = \{\forall i, (\nu y_i) = 0\}$ , then apply these proofs along  $\mathcal{T}$ , to each of the (real) coefficients  $f_a$ , factor of  $\nu^a$ , of the nilomorphic functions  $f$  that appear. As the latter are adapted along  $\mathcal{T}$  (see 2.15), this means applying the proofs on  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$  for each  $f_a$ . Then extend the value of all functions along the leaves of  $\mathcal{I}$  by the formula of Proposition 2.16. So, you get the announced potentials  $u$ .

**Part (b) for cases (1), (2) and (2')** — hence also for their complexifications  $(1^{\mathbb{C}})$  and  $(2^{\mathbb{C}})$ . We adapt the standard arguments given in Proposition 1.27. First, we take nilomorphic coordinates being, at the origin in  $\mathcal{T} = \{(\nu y) = 0\}$ , tangent to normal coordinates *i.e.* such that the  $X_i.g(X_j, X_k)$  vanish; in particular all  $D_{X_i}X_j$  are null. As  $DN = 0$ , so are the  $D_{N^b X_i}N^a X_j$ , and we still get, for any vectors  $A, B, U, V$  among the  $N^a X_i$  at the origin:

$$g(R(A, B)U, V) = \frac{1}{2}(A.U.(g(B, V)) - B.U.(g(A, V)) - A.V.(g(B, U)) + B.V.(g(A, U))).$$

As  $N$  is also self adjoint,  $R(N^a A, B) = R(A, N^a B)$  for all  $a$ . The  $R(X_i, N^a X_j)$  are determined by their restriction on  $T\mathcal{T}$  *i.e.* by the  $R(X_i, N^a X_j)X_k$ . We denote by  $\widehat{R}$  this restriction.

In case (1), for each  $a$ ,  $\widehat{R}(X_i, N^a X_j)$ , which is defined at the origin on  $T\mathcal{T}/\ker N^a$ , is the alternate part of the bilinear form:

$$\beta_{i,j,a} : (U, V) \mapsto X_i.U.g(N^a X_j, V) - X_j.U.g(N^a X_i, V),$$

also defined on  $T\mathcal{T}/\ker N^a$ . For each  $a$ , the  $\beta_{i,j,a}$  depend on the second derivatives at 0 of the coefficients of  $g(\cdot, N^a \cdot)$ , defined on  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)$ . Those derivatives are free in normal coordinates. Indeed  $g(\cdot, N^a \cdot)$  is the coefficient of  $\nu^{n-1-a}$  of the nilomorphic metric  $h$  given in Theorem 2.31, hence is chosen freely. So, on a dense open subset of the 2-jets of metrics, the alternate parts of the  $(\beta_{i,j,a})_{i,j=1}^D$  are linearly independent and hence span a  $\frac{\delta(\delta-1)}{2}$ -dimensional space, with  $\delta = \dim(\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)) = \#\{i; N^a X_i \neq 0\}$ . So, the sum for all  $a$  of those dimensions is the number  $K$  of triples  $(i, j, a)$  with  $i < j$  and  $N^a X_j \neq 0$ , and generically, the holonomy algebra is  $K$ -dimensional. Now an element  $\gamma$  of  $\mathfrak{o}_d(\mathbb{R})^N$  is precisely given by the  $g(\gamma(N^a X_j), X_i)$  for those triples  $(i, j, a)$  *i.e.*  $\dim \mathfrak{o}_d(\mathbb{R})^N = K$ . We are done.

The adaptation for case (2) is similar. The forms  $\beta_{i,j,a}$  appearing are:

$$\beta_{i,j,a} : (Z_k, \overline{Z}_l) \mapsto \frac{1}{2}(-\overline{Z}_j.Z_k.(g(Z_i, N^a \overline{Z}_l)) - Z_i.\overline{Z}_l.(g(\overline{Z}_j, N^a Z_k))),$$

defined for  $i, j$  in  $\llbracket 1, \frac{D}{2} \rrbracket$  and  $a \in \llbracket 0, n-1 \rrbracket$ , see the proof of Proposition 1.27. Each  $\beta_{i,j,a}$  is given by the derivatives of  $u_{n-1-a}$ , the coefficient of  $\nu^{n-1-a}$  in the potential  $u$  given by part (a) of the theorem. This  $u_{n-1-a}$  is defined on  $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)$ , so the  $(\beta_{i,j,a})_{i,j}$  may be chosen freely and span a  $(\frac{\delta}{2})^2$ -dimensional space, with  $\delta = \dim(\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)) = \#\{i; N^a X_i \neq 0\}$ . The sum for all  $a$  of those dimensions is the number  $K'$  of triples  $(i, j, a)$  with  $i < j \leq \frac{D}{2}$  and  $N^a Z_j \neq 0$ . So generically, the holonomy algebra is  $K'$ -dimensional. Now an element  $\gamma$  of  $\mathfrak{u}_{d/2}^N$  is precisely given by the  $g(\gamma(N^a Z_j), \overline{Z}_j)$  for those triples  $(i, j, a)$  *i.e.*  $\dim \mathfrak{u}_{d/2}^N = K'$ . We are done. Case (2') is entirely similar and left to the reader.  $\square$

**3.9 Remark** Take  $g$  a metric on  $\mathbb{R}^D$  having a holonomy group  $H_0$  of one of the eight types listed in Remark 1.15. In particular,  $\text{End}(T\mathcal{M})^{\mathfrak{h}_0} = \mathfrak{s}$  is semi-simple, and of any of the

eight types of Theorem 1.11. If  $g$  is *real analytic*, then around the origin, in coordinates  $(x_i)_{i=1}^D$ , each of its coefficients  $g_{i,j}$  is a power series of the form:  $\sum_{\alpha} g_{i,j,\alpha} x^{\alpha}$  where  $\alpha = (\alpha_1, \dots, \alpha_D) \in \mathbb{N}^D$  is a multi-index. Tensorise the space  $\mathbb{R}^D$  by  $\mathbb{R}[\nu] \simeq \mathbb{R}[X]/(X^n)$ , obtaining  $(E, N) := \mathbb{R}[\nu]^D$ . Each variable  $x_i$  becomes  $z_i := x_i + (\nu y_i)$  and the expansions of the  $g_{i,j}$  become  $h_{i,j} = \sum_{\alpha} g_{i,j,\alpha} z^{\alpha}$ , which are nilomorphic by construction. We let the reader check that the real metric associated with this nilomorphic metric  $h$  has  $H_0 \otimes \mathbb{R}[\nu] \simeq \widehat{H}_0^N$  as holonomy group, with  $\widehat{H}_0$  the group of the type of  $H_0$  in the table of Remark 1.15, acting on  $\mathbb{R}^{nD}$  instead of  $\mathbb{R}^D$ . Therefore,  $g$  has  $\mathfrak{s} \otimes \mathbb{R}[\nu] \simeq (\mathfrak{s} \otimes \mathbb{R}^n) \times \langle N \rangle$  as algebra of parallel endomorphisms. By this means, as in Theorem 3.2, we build a metric with  $\widehat{H}_0^N$  as holonomy group, with  $N$  in the bicommutant of  $\widehat{H}_0$ , if all the invariant factors of  $N$  have the same degree  $n$ . So this works like a complexification.

Yet this does not parametrise the set of all such metrics. Notice indeed that the restriction of  $h$  to  $\{(\nu y) = 0\}$ , *i.e.*  $h$  applied to tangent vectors to  $\{(\nu y) = 0\}$ , has value in  $\mathbb{R}$ .

In fact, we may proceed similarly for a nilpotent endomorphism  $N$  of any similarity type. It is however a bit more cumbersome, so we present it apart, moreover we let the reader check the proofs. Consider a metric  $g$  like above. For each  $i$  set, formally,  $z_i := x_i + (\nu y_i)$  and choose some  $n(i) \in \llbracket 1, n \rrbracket$ . The  $\nu^{n-n(i)} z_i$  parametrise an  $\mathbb{R}[\nu]$ -module  $(E, N) := \prod_{i=1}^D (\nu^{n-n(i)} \mathbb{R}[\nu])$ , the original  $\mathbb{R}^D$  being identified with the level  $\{(\nu y) = 0\}$ . Denote by  $d$  the real dimension of  $E$ . Suppose additionally that:

- the  $n(i)$  are chosen so that the induced flag  $\ker N \subset \ker N^2 \subset \dots \subset \ker N^{n-1} \subset E$  is  $\mathfrak{s}$ -stable,
- the  $g_{i,j,\alpha}$  are such that the  $\mathbb{R}[\nu]$ -valued form:

$$h := \sum_{i,j=1}^D \sum_{\alpha} g_{i,j,\alpha} \nu^{n-n(\alpha)} z^{\alpha} dz_i \otimes dz_j \quad \text{with } n(\alpha) := \min_{\{i; \alpha_i \neq 0\}} \{n(i)\}$$

makes sense and is nilomorphic, *i.e.* that the value  $\check{h}$  of  $h$  along  $\{(\nu y) = 0\}$  is adapted (see Definition 2.15). We let the reader check that it means:  $g_{i,j,\alpha} = 0$  as soon as  $n(i) < n(\alpha)$  or  $n(j) < n(\alpha)$ .

Denote by  $\mathbf{K}$  the trace on  $\{(\nu y) = 0\}$  of the flag of the of the  $\ker N^a$ . Then, among the sequences  $(g_{i,j,\alpha})_{i,j,\alpha}$  such that the holonomy group of the corresponding metric  $g$  is some  $H_0$  in the list of Remark 1.15, these additional constraints induce metrics the holonomy group of which is  $H'_0 := \{\gamma \in H_0; \gamma(\mathbf{K}) = \mathbf{K}\}$ . Moreover, the real metric associated with  $h$  has  $\widehat{H}_0^N$  as holonomy group, with  $\widehat{H}_0$  the group of the same type as  $H_0$  in the table of Remark 1.15, but acting on  $\mathbb{R}^d$ . This gives also the existence part of Theorem 3.2, but does not parametrise the whole set of metrics sharing this holonomy group, for the same reason as above: the value  $\check{h}$  of  $h$  along  $\{(\nu y) = 0\}$  is such that each  $\check{h}_a$  is given by a matrix of the form  $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & 0 \end{pmatrix}$ , as given in Remark 2.41. This is not the case for a generic metric with this holonomy group.

**3.10 Remark** The note p. 41 lets the reader notice that the technique used in the proof of Theorem 3.2 may be used as a standard way to generalise reasonings on germs of real functions to germs of nilomorphic ones. In particular, by this means, or as in Remark 3.9 above, for the analytic category, we might show similar statements as Theorem 3.2 for  $H$  any semi-simple classical pseudo-Riemannian holonomy group.

## 4 A glimpse on the case where $\mathfrak{n}$ is not principal

We investigate here the simplest example in which  $\mathfrak{n} = \text{Rad}(\text{End}(T\mathcal{M})^N)$  does not admit a unique generator. This will show a type of phenomenon appearing when  $\mathfrak{n}$  is abelian, non principal. We will see that the results of both Parts 1 and 2 are needed to describe it.

This case is the following. We parametrise the set  $\mathcal{G}$  of germs of metrics such that  $\mathfrak{n} = \langle N, N' \rangle$  with  $N^2 = N'^2 = NN' = N'N = 0$ , and  $N$  and  $N'$  self adjoint. To simplify a bit more, we also assume that  $\text{Im } N = \ker N$  — our goal here is no kind of general theory.

**4.1 Remark (i)** Then, there is an  $U \in \text{End}(T\mathcal{M}/\text{Im } N)$  such that  $N' = NU$ , which makes sense as the argument of  $N$  may be defined only modulo  $\ker N$ . Indeed,  $N$  gives an isomorphism  $\theta_N : T\mathcal{M}/\ker N \rightarrow \ker N$ . As  $NN' = N'N = 0$ ,  $N'$  induces also a morphism  $\theta_{N'} : T\mathcal{M}/\ker N \rightarrow \ker N$ , and is determined by it. Set  $U := \theta_N^{-1} \circ \theta_{N'} \in \text{End}(T\mathcal{M}/\ker N)$ .

(ii) Identifying  $U$  with its matrix, in coordinates  $(y_i, x_i)_{i=1}^{d/2}$  adapted to  $N$ :

$$\text{Mat}(N) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{Mat}(N') = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}.$$

Then, using Lemma 3.1 complemented by Remark 3.5, we may take coordinates such that the matrix of  $U$  is also constant — and for instance in Jordan form. Explicitly,  $U$  is well defined on  $\mathcal{M}/\mathcal{I}$  (in other words it is  $\mathcal{I}$ -basic) and, if  $\prod_{\alpha} P_{\alpha}^{n_{\alpha}}$  is the decomposition of its minimal polynomial in powers of irreducible polynomials, we may take coordinates  $(x_i)_i$  which are:

- product coordinates for  $\mathcal{M}/\mathcal{I} \simeq \prod_k \overline{\mathcal{M}}_k$ , integration of the decomposition  $\bigoplus_{\alpha}^{\perp} \ker P_{\alpha}^{n_{\alpha}}$ ,
- on each factor, adapted to the nilpotent part of  $U$  on it and, if  $\deg P_{\alpha} = 2$  i.e. if the semi-simple part of  $U$  induces a complex structure  $\underline{J}_{\alpha}$  on it, also complex coordinates for it.

**4.2 Proposition** *With the  $U \in \text{End}(T\mathcal{M}/\text{Im } N)$  introduced in Remark 4.1 (i), a metric  $g$  makes  $N$  and  $N'$  parallel if and only if:*

- it is the real metric  $h_1$  associated with a  $(\nu, N)$ -nilomorphic metric  $h = h_0 + \nu h_1$  with value in  $\mathbb{R}[\nu] = \mathbb{R}[X]/(X^2)$ ,
- the bilinear form  $h_0$ , defined on  $\mathcal{M}/\mathcal{I}$ , makes  $U$  parallel (recall that  $h_0 = g(\cdot, N \cdot)$  so is nondegenerate, hence is a (pseudo-)Riemannian metric, on  $\mathcal{M}/\mathcal{I}$ ).

Then there exist coordinates  $(x_i, y_i)_i$  simultaneously  $N$ - and  $N'$ -adapted, as introduced in Remark 4.1 (ii). In such coordinates, on each factor  $\overline{\mathcal{M}}_k$ ,  $h_0$  is itself of the form given by Theorem 2.31, for the nilpotent part  $N_k$  of  $U$  on  $\overline{\mathcal{M}}_k$  (i.e.  $h_0$  is the real metric associated with some  $(\mu, N_k)$ -nilomorphic metric, with  $\mathbb{R}[\mu] = \mathbb{R}[X]/(X^{n_k})$ ), and also complex Riemannian for  $\underline{J}_k$  if  $\deg P_k = 2$  (then  $\mathbb{R}[\mu]$  is replaced by  $\mathbb{C}[\mu]$ ).

**Proof.** After Theorem 2.31,  $g$  makes  $N$  parallel if and only if it satisfies the first point. It makes also  $N'$  parallel if and only if for any  $(i, j, k)$ ,  $g(D_{X_i} N' X_j, X_k) = g(D_{X_i} X_j, N' X_k)$ , i.e.  $g(D_{X_i} N U X_j, X_k) = g(D_{X_i} X_j, N U X_k)$ , or  $g(D_{X_i} U X_j, N X_k) = g(D_{X_i} X_j, N U X_k)$ , as  $DN = 0$ . This means  $h_0(D_{X_i} U X_j, X_k) = h_0(D_{X_i} X_j, U X_k)$  i.e. that  $U$  is parallel for  $h_0$ .  $\square$

In real terms, following Example 2.38,  $\text{Mat}_{((Y_i)_i, (X_i)_i)}(g) = \begin{pmatrix} 0 & G^0 \\ G^0 & G^1 \end{pmatrix}$  where:

- $G^0$ , depending only on the  $(x_i)_i$ , is the matrix of a metric on  $\mathcal{M}/\mathcal{I}$  making  $U$  parallel,

$-G^1 = B^1 + \sum_i (\frac{\partial G^1}{\partial x_i}) y_{i,1}$ , with  $B^1$  the matrix of any bilinear symmetric form, depending only on the  $(x_i)_i$ .

So building a metric making  $N$  and  $N'$  parallel means taking a metric such that  $N$  is, and adding constraints on it by repeating the general story of this article, on the quotient  $\mathcal{M}/\mathcal{I}$ , endowed with the metric  $g(\cdot, N \cdot)$  and of a parallel endomorphism  $U$  induced by  $N'$ .

If  $(N, N')$  is a general pair of commuting nilpotent endomorphisms, their characteristic flags of subspaces  $F^{a,b}$  introduced at the beginning of §2.1 induce, each alone and with each other, a lot of quotient spaces  $E_\alpha$  on each of which the metric  $g$  and the pair  $(N, N')$  induce an endomorphism  $U_\alpha$  and a metric  $g_\alpha$ . Building a metric making  $N$  and  $N'$  parallel means in particular making each  $U_\alpha$  parallel for  $g_\alpha$ , so repeating Theorem 2.31 on  $E_\alpha$ , applied to some coefficient of some nilomorphic metric, sometimes several times successively. In fact, the situation is more complicated, as a pair of commuting nilpotent endomorphisms is not something simple — *e.g.* think simply to the case treated in Proposition 4.2, without the assumption  $\text{Im } N = \ker N$ . Finally, if more than two endomorphisms  $N$  and  $N'$  are involved, it may appear quotients of the type of the  $E_\alpha$  on which any of the situations of Theorem 1.11 appears, even if  $\mathfrak{s} = \mathbb{R} \text{Id}$  on  $T\mathcal{M}$  itself.

So the general situation, though not entirely new with respect to the case where  $\mathfrak{n}$  is principal, seems to be complicated. What the good next questions are about it is still unclear. The case where  $\mathfrak{n}$  is not abelian may induce new phenomena, but is strongly constrained, when  $\mathfrak{n}$  consists of self adjoint elements, by Proposition 1.8.

## 5 Parallel endomorphisms and Ricci curvature

The Ricci form  $\text{ric}(\cdot, J \cdot)$  has remarkable properties on Kähler manifolds. Let us determine the properties of the corresponding forms when  $g$  admits other parallel endomorphism fields than a skew symmetric complex structure.

**5.1 Theorem** *Suppose that  $U$  is a parallel endomorphism field for some pseudo-Riemannian metric  $g$ ;  $a$  and  $b$  denote any two tangent vectors at some point.*

(i) *If  $U$  is self adjoint:*

- a)  $\text{ric}(a, Ub) = \text{ric}(Ua, b) = \text{tr}(U(R(a, \cdot)b))$  (and  $U$  and  $R(a, \cdot)b$  commute),
- b) (standard result) if  $U = \underline{J}$  is a complex structure,  $g$  is the real part of the  $\underline{J}$ -complex metric  $g_{\mathbb{C}} := g(\cdot, \cdot) - i g(\cdot, \underline{J} \cdot)$ , and the Ricci curvature of  $g_{\mathbb{C}}$  is  $\text{ric}_{\mathbb{C}} = \text{ric}(\cdot, \cdot) - i \text{ric}(\cdot, \underline{J} \cdot)$ ,
- c) if  $U = N \neq 0$  is nilpotent, then  $\text{ric}$  is degenerate and:  $\text{Im } N \subset \ker \text{ric}$ .

(ii) *If  $U$  is skew adjoint:*

- a)  $\text{ric}(a, Ub) = -\text{ric}(Ua, b) = \frac{1}{2} \text{tr}(U \circ R(a, b))$ ,
- b) if  $U = N \neq 0$  is nilpotent, then  $\text{ric}$  is degenerate and:  $\text{Im } N \subset \ker \text{ric}$ ,
- c) if  $V$  is another skew symmetric parallel endomorphism with  $VU = -UV$ , and if  $U$  and  $V$  are invertible, then  $\text{ric} = 0$ . So (standard result) cases (3), (3'), (3<sup>C</sup>) of Theorem 1.11 are Ricci-flat.

**Proof.** Take  $U$  self adjoint, then the whole of **a)** follows from Remark 1.10. Point **b)** is standard and for **c)**, after **a)**,  $\text{ric}(a, Nb) = \text{tr}(N(R(a, \cdot)b))$ , and as  $N$  and  $R(a, \cdot)b$  commute, their product is also nilpotent, so trace free.

Now take  $U$  skew adjoint.

$$\begin{aligned}
\text{ric}(a, Ub) &= \text{tr}(R(a, \cdot)Ub) \\
&= \text{tr}(U(R(a, \cdot)b)) \quad \text{as } U, \text{ being parallel, commutes with } R(a, \cdot), \\
&= \text{tr}(R(a, U\cdot)b) \quad \text{as } \text{tr}(UV) = \text{tr}(VU), \\
&= -\text{tr}(R(Ua, \cdot)b).
\end{aligned}$$

For the last line, take any  $u, v, w$ :  $g(R(Ua, u)v, w) = g(R(v, w)Ua, u) = g(UR(v, w)a, u) = -g(R(v, w)a, Uu) = -g(R(a, Uu)v, w)$ . So finally,  $\text{ric}(a, Ub) = -\text{ric}(Ua, b)$ . Besides:

$$\begin{aligned}
\text{ric}(Ua, b) &= \text{ric}(b, Ua) \\
&= \text{tr}(U(R(b, \cdot)a)) \\
&= -\text{tr}(U(R(\cdot, a)b)) - \text{tr}(U(R(a, b)\cdot)) \quad \text{by the Bianchi identity,} \\
&= \text{tr}(R(a, \cdot)Ub) - \text{tr}(U \circ R(a, b)) \quad \text{as } U \text{ commutes with } R(a, \cdot) = -R(\cdot, a), \\
&= \text{ric}(a, Ub) - \text{tr}(U \circ R(a, b)).
\end{aligned}$$

As  $\text{ric}(Ua, b) = -\text{ric}(a, Ub)$ , we get **a**). Point **b**) follows:  $\text{ric}(a, Nb) = \frac{1}{2} \text{tr}(N \circ R(a, b)) = 0$  as  $N \circ R(a, b) = R(a, b) \circ N$  is nilpotent. Point **c**) is only a way to re-find that  $\text{ric} = 0$  in cases **(3)**, **(3')**, **(3<sup>C</sup>)**, using **a**). Indeed, if  $U$  and  $V$  are as announced, any  $b$  can be written  $b = UVc$ , and:  $\text{ric}(a, b) = \text{ric}(a, UVc) = -\text{ric}(Ua, Vc) = -\frac{1}{2} \text{tr}(V \circ R(Ua, c)) = \frac{1}{2} \text{tr}(V \circ R(a, Uc)) = \text{ric}(a, VUc) = -\text{ric}(a, UVc) = -\text{ric}(a, b)$ .  $\square$

**5.2 Corollary** *Let us denote by Ric the Ricci endomorphism, defined by  $\text{ric} = g(\cdot, \text{Ric} \cdot)$ . If the metric is indecomposable (in a local Riemannian product) and such that  $\text{ric}$  is parallel, then Ric is either semi-simple or 2-step nilpotent.*

**Proof.** As  $g$  is indecomposable, the minimal polynomial of Ric is of the form  $P^\alpha$  with  $P$  irreducible, see *Claim 1* on p. 7 in the proof of Theorem 1.11. So Ric is either invertible or nilpotent. Apply Theorem 5.1 **(i) c)** to the nilpotent part  $N_{\text{Ric}}$  of Ric: if Ric is invertible,  $\ker \text{ric} = \{0\}$  so  $N_{\text{Ric}} = 0$ , else  $\text{Ric}^2 = N_{\text{Ric}}^2 = 0$ . We re-find here the result of [7].  $\square$

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